Beside the Iterable Unit: Reply to Steffe et al.

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Abstract

In this volume, Steffe, Liss, and Lee discuss a sequence of reorganizations and accommodations that construct schemes of intensive quantity from numerical schemes. In this sequence, the ideas of "iterable unit" and "partitioning" feature prominently. However, Steffe et al. also acknowledge some limitations of their sequence, in particular in dealing with notions of "all comparisons" and "any but no particular" state. In this reply, I use data from two high school students and a high school teacher to propose that ideas of "any" and particularly "all" states are built from a yet to be defined system of "smooth" schemes based in forming images of change in experiential time that is separate from but coordinated with "chunky" schemes based in iteration and partitioning.

Beside the Iterable Unit: Reply to Steffe et al.

Steffe Liss and Lee's paper in this volume (Steffe, Liss II, & Lee, in-press) is an excellent illustration of the necessity of a distributive partitioning scheme in the construction of intensive quantity. Steffe et al. also say however that this scheme cannot be sufficient: "But in no way would such a scheme be sufficient in the construction of intensive quantity, because a concept of intensive quantity seemingly involves not two fixed states, but a variable state that symbolizes all comparisons" (p. 27). In this reply I discuss the idea that a distributive partitioning scheme, although necessary, is not sufficient, and point out areas that serve as opportunities for future research.

The theme in this reply will be the problem identified by Steffe et al. above: constructing a variable state that symbolizes all comparisons. The problems of "variable" and "all" or "any" state come up repeatedly in Steffe's paper, particularly when discussing the transition from a distributive partitioning scheme to a recursive distributive partitioning where "that *any but no particular* partition can be conceptualized to establish [equivalent] ratios" (pp. 18, emphasis added), and the related idea of an extensive quantitative variable as "the potential result of measuring a *varying* quantity at *any but no particular* time" (p. 4, emphasis in original). In this reply I will propose the idea that these notions of "all" or "any" depends on images of continuous variation constructed separately from the schemes Steffe describes. In particular, I argue that that these images depend on the *absence* of an iterable unit, and the images are used in coordination with a separate system of schemes built from the iterable unit that Steffe describes, hence the title "beside the iterable unit."

Chunky and Smooth

Before we begin, I need to build some definitions. By "image," I mean image in the sense that Thompson & Thompson (1992) describe as imagined or anticipated actions on an object. Thompson & Thompson attribute three types of images to Piaget (1967): briefly summarized as (i) image as object[1], (ii) image as state, and (iii) image as transformations. Of particular interest are the latter two of the three types of images that Thompson & Thompson attribute to Piaget: image as an outcome of an action performed on an object (image as state), or as a fully dynamic and mobile image in which the transformations, not the object or its states, are the focus (image as transformations).

Two other terms that I wish to define are "chunky" and "smooth" and their companion terms "chunk," "chunky/smooth perception," "chunky/smooth reasoning," and "chunky/smooth variation." These terms emerged as ways of making sense of the behavior of students working in dynamical systems (Castillo-Garsow, 2010, 2012) and as ways to characterize the distinctions between the Confrey (Confrey & Smith, 1994, 1995) and Thompson (Saldanha & Thompson, 1998; Thompson, 2002, 2008a) styles of covariation The distinction between "chunky" and "smooth" is inspired in part by the distinction between path-semantics and state-change semantics in linguistic studies of time (Bohnemeyer, 2010)

By *chunky* I mean involving a completed change. *Chunky reasoning* is forming an image of completed change, analogous to image as state described above. *Chunky variation* is forming an image of completed change of a numerical object such as a quantity or variable.

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1 This is a particularly poor characterization of the nuances of the first type of image, but it will suffice for the present discussion, and I invite you to read the cited papers for details

By *smooth* I mean involving a change in progress. *Smooth reasoning* is forming an image of dynamic change in progress, analogous to the image as transformations described above. *Smooth variation* is forming an image of change in progress of a numerical object such as a quantity or variable.

As a metaphor: consider an animated film. An animator drawing the film would perceive each frame as a state of the characters in the frame. Characters change location and position from frame to frame without passing intervening states. However when the film is watched, these individual frames create an illusion of motion, where now the character is imagined to pass through the intervening space. If the animator were to break this illusion by, for example, having a character appear to wave their arm through a solid wall, the viewers would see that as exception. This example of the animator vs. the movie viewer may be thought of as chunky perception vs. smooth perception. Chunky vs. smooth reasoning may be thought of as constructing an image (in the above sense) of a dynamic situation. If we imagine a bungee jumper jumping off of a bridge, we can imagine viewing the jump as a series of frames or snapshots (chunky), or we can imagine viewing a jump as a played movie or an event in progress --- one that is changing continuously as we experience imagining it (smooth). Chunky or smooth variation would be when we take a quantity that we can measure about the situation and apply it to the image. For example, in the chunky image of the bungee jumper, we might imaging measuring time as the number of frames of the movie that have passed. In the smooth image of the bungee jumper, we might imagine time as a running stopwatch counting the number of seconds that have passed since the bungee jumper jumped off the bridge, passing through every real valued number of seconds in-between; however this example -

-- used to highlight the differences between chunky and smooth --- is not an accurate characterization of pure smooth variation, which I will discuss in more detail later.

A *chunk* is defined an imagined or anticipate completed change. A chunk may become an iterable unit. For students who have the types of partitioning schemes Steffe describes, chunks can be subdivided. At three levels of units, the result of a subdivided chunk is another chunk, or as Steffe et al. put it "One-seventh is said to be freed from the unit segment of which it is a part and so it can be used as an iterable unit in the sense that the unit of one is an iterable unit" (p. 13, this volume). At two levels of units such as the splitting scheme, Steffe et al. also describe iterating behavior as part of reconstructing the imagined partition, and so I would also characterize this type of incomplete iterable unit as a chunk.

Pure smooth reasoning (and pure smooth variation) occurs in the absence of chunks. As such, the example of counting seconds above is not an example of pure smooth variation, because the stopwatch is imagined to be marking off units of 1. Rather, this can be thought of as an example of a type of "hybrid" variation, in which smooth variation (change in progress) generates a chunk (iterable unit seconds). This "using smooth to consider chunks[2]" can be thought of as the initial stage of the hybrid variation described by Thompson (Saldanha & Thompson, 1998; Thompson, 2008b, 2013) prior to engaging in recursive partitioning.

Because it must occur in the absence of chunks, pure smooth variation is qualitative in the sense that it is numerical without attention to the actual numbers generated. This might occur if, as we imagine the bungee jumper, we imagine that we

[2] Thanks go to Kevin Moore for this name

could measure the time since the jump, and that it could be measured in a unit such as seconds, and that as the jump progresses, that number is getting larger, but at no point do we imagine what that value actually is. This image can be likened to turning around a stopwatch so that we cannot read the display and the particular numerical values. We imagine time passing the numbers increasing without giving particular preference to a single unit such as seconds over any other unit such as minutes or Planck time. No matter the choice of unit, time is passing and the number in that unit is increasing, and that is sufficient.

The finger exercise described in Thompson's didactic objects paper (Lima, McClain, Castillo-Garsow, & Thompson, 2009; P. W. Thompson, 2002) provides another example of both categories of variation. Depending on how it is implemented, it might be thought of as an exercise in moving fingers to show increasing and/or decreasing quantities (pure smooth variation) or an exercise in moving fingers along a number line that has a beginning, and end, and increments (hybrid variation).

Chunky and smooth should not be thought of as schemes in the sense that Steffe et al. use the term. They are simply not precise enough, and this paper should only be thought of as laying the groundwork for future research in this area. It may be helpful to think of chunky and smooth as categories used to describe schemes. For example, the schemes described by Steffe et al. in the paper fall in the chunky and hybrid categories. As categories, chunky and smooth are not mutually exclusive (as seen in the hybrid variation examples), and certainly not exhaustive. They may not even exhaust the space of schemes of change.

In brief, *chunky variation* may be thought of as a discrete and quantitative image of a dynamic situation based on an iterable unit, while *smooth variation* may be thought of as a continuous and qualitative image of a dynamic situation built with yet to be characterized tools other than the iterable unit.

Smooth, Chunky and Student Development

Chunky and smooth reasoning begin early in life. Pure smooth reasoning is demonstrated in children at least as early as object permanence (Glasersfeld, 1996; Piaget, 1954). When an infant tracks a moving object hidden behind a screen, the infant is engaging in pure smooth reasoning. Purely at the level of sensorimotor intelligence, he/she tracks the object's motion in a way that is anticipatory, transformative and updated in experiential time.

In the experiment with water flowing from one bottle to another from *Child's conception of time*, Piaget (1969) describes a child, Ber, in the process of developing using smooth to consider chunky reasoning. Ber is able to trace the flow of water with his/her finger (a "kinetic interpretation of the flow process as a whole" (Piaget, 1969, p. 15), and uses this image to order pairs of states (chunks) of the bottle, but is unable to reconstruct the entire sequence of states. However, despite the progression from smooth sensorimotor reasoning to chunky operational reasoning described in the bottle experiment, it would be a mistake to claim that chunky reasoning as a whole is more advanced or more desirable than smooth reasoning. It is merely more rigorous.

One looking at the beginning and ending of mathematics classes might miss the importance of smooth reasoning to mathematics. In the beginning, as students are learning to count and partition, chunks play an incredibly important role (Steffe et al., in-press). At the end of mathematics classes (upper undergraduate), when students are learning to formalize ideas of number and limit, the arbitrarily small but not infinitesimal chunks ε and δ again become incredibly important.

But there is a critically important stage in schooling, starting somewhere around pre-calculus and potentially extending to differential equations (depending on the student) when students must be able to distinguish the discrete and continuous, between a sequence and its limit, between the presence and absence of infinitesimal holes. As the teaching experiment in the next section will show, there are fundamental problems at this level where --- without the formal, chunky tools of epsilons and deltas --- the only way that students can make sense of these distinctions is by drawing upon tools of smooth reasoning.

And far from being obsoleted by chunky reasoning, these smooth reasoning tools are ones that many professional mathematicians continue to use every day, as continuous dynamical systems such as differential equations remain an area of active research. This is not meant to imply that every student should or will learn differential equations. It is only meant to give a proof of concept example of how a neglected idea in traditional schooling (smooth reasoning) has an important mathematical use. Rather I suggest that smooth reasoning is critical to the forming the initial image (corresponding to Carlson & Bloom's (2005) orienting phase) of dynamical systems from at least the Algebra I & II level. One example of these types of systems is the parametric function activities described by Bishop and John (2008), in which an image of point moving along a path of *all* its possible values plays a critical role.

The Teaching Experiment

This section will focus on the role of chunky and smooth in developing understandings of intensive quantities in the form of rates. I will use data from "Derek" and "Tiffany," two Algebra II students who participated in a teaching experiment in basic differential equations.

Tiffany & Derek were high performing non-honors Algebra II students, who agreed to participate in a Steffe and Thompson (2000) style teaching experiment that followed Thompson's (2008a) construction of compound interest with the goal of teaching students the rate-proportional-to-amount property of exponential growth used in differential equations. The teaching experiment began with simple interest, and progressed through piecewise linear compound interest before introducing continuous compounding through the perspective of constant per-capita rate of change (Castillo-Garsow, 2010).

Under this structure, simple interest is imagined as a family of linear functions in which the dollar per year rate of change of the line (the slope) is proportional to the initial value (dollars) of the function. Derek and Tiffany characterized this sort of simple interest relationship with the equation $q(x,n)=n+.08(n)x$, where n is the initial number of dollars invested, x is the number of years since investment, and 0.08 corresponds to the 8% interest rate[3] of the account.

Compound interest was developed as an extension of simple interest. The idea was to introduce a competing bank that automatically reinvested (compounded) an

3 The use of the word "rate" here is in the financial sense and not the mathematical sense. Although it is possible to connect this meaning of rate to the slope of a linear function, the students were not aware of how to do this. In keeping with mathematical tradition henceforth, "rate" will always refer to a linear rate of growth such as dollars per year, and never to interest rate.

investment every quarter of a year so that the dollar per year rate (slope) was no longer proportional to initial amount, but instead proportional to the amount at the beginning of the quarter. The students individually investigated this task for an initial investment of $500. Derek in particular imagined that over the first quarter of the year, the account grew linearly using the function $q(x,500)=500+.08(500)x$, and that over the second quarter, the account grew linearly but with a new dollar per year rate (a new slope) calculated from the amount in the account at the end of the first quarter (Figure 1). Tiffany's work was similar, although she focused much more on how to calculate the value at the ends of the quarters, rather than on imagining the account growing linearly in-between. Both students completed equations describing the behavior of the account (Figure 2).

[Figure 1 about here]

[Figure 2 about here]

Following this investigation of compound interest, the students investigated phase plane representations of these accounts, in which instead of graphing account value with respect to time, the students graphed account rate (dollar per year growth) with respect to account value (Figure 3).

[Figure 3 about here]

After observing a linear relationship in the initial points of phase plane step function for compound interest, the students were asked to consider what would happen if compounding was happening all the time (Figure 4), and lastly to recreate a graph of account value with respect to time from the phase plane graph of continuous compounding. What follows is a description of both students based on data across the teaching experiment, with the goal of explaining the dramatic differences in their abilities to complete this final task.

[Figure 4 about here]

Tiffany's partitioning

The overall story of Tiffany, when viewed through the lens of Steffe's work, might be interpreted as an unsuccessful attempt to move Tiffany to recursive distributive partitioning, and when viewed from this perspective, it may be a useful story for identifying the critical ideas behind extending distributive partitioning to "any" or "all" partitions.

Tiffany was a student who had a functional partitive fraction scheme (see Castillo-Garsow (2010) for a full treatment of Tiffany), but she greatly preferred to use iterative whole number schemes, as seen in Table 1.

[Table 1 about here]

On line 1 of Table 1, Tiffany appears to have claimed that three twelfths is a truer depiction of a quarter of a year than one quarter. By way of contrast lines 3 & 4, give us further insight into Tiffany's reasoning on Line 1. Although she used the words "one twelfth" she initially imagined "one month." It is not until lines 3&4 that she imagines one-quarter measured in a non-standard unit. Imagining one-twelfth as one month enabled her to recast time in terms of a new unit "months" and iterate those months to form a quarter as a whole number of month-sized chunks called one-twelfth, instead of as a piece of a year-sized chunk. This is why Tiffany said "three twelfths" was more "literally" a quarter of a year than one-fourth (Line 1). Although Tiffany was capable of partitioning a year into parts and keeping both the whole and the part in mind (lines 3-4), Tiffany used this approach of recasting a fraction as one in a different unit whenever she could, disregarding the year and operating at one and two levels of units whenever possible[4].

Tiffany and speed

In an attempt to push Tiffany towards using partitioning more often (using one quarter of a year at three levels of units rather than counting quarters, months, or days and dropping the year), the researchers used a context with which Tiffany was more familiar: speed. These discussions took on the same character as those in the "Over and Back" papers (P. W. Thompson, 1994; P. W. Thompson & Thompson, 1994; A. G. Thompson & Thompson, 1996), in that Tiffany initially took as speed-length perspective and was pushed to partition.

[Table 2 about here]

The question in line 3 Table 2 could be interpreted in a number of ways: For example, an number of faculty I have presented this to assumed Tiffany understood the question as "Is it possible to make a trip that consists entirely of starting at a stop, traveling one second at sixty-five miles per hour and then immediately stopping again?" However, this is not the way the question was intended, and it is not the way that Tiffany interpreted it. Rather Tiffany's "no" on line 4 is a claim that it is impossible to travel at sixty-five miles per hour for any time period less than an hour --- indicating that she was thinking in terms of iterated speed lengths. The "you would have to do" on line 4 is a reference to the partitioning strategy she was being taught.

[4] Although Tiffany did occasionally use improper fractions and sometimes appeared to have an iterative fraction scheme and three levels of units, her later difficulties suggest that in fact, she was just really good at changing units and was actually taking on three levels of units two at a time

What is particularly interesting about this example over the speed-length/partitioning examples in previous studies (P. W. Thompson, 1994; P. W. Thompson & Thompson, 1994; A. G. Thompson & Thompson, 1996) is that unlike the students in those studies, Tiffany is old enough to drive, and drives herself to school everyday. However, she does not answer Pat's question based on her driving experience, but instead based on the mathematical methods that she has been taught. For Tiffany in the context of a math class, it is impossible to travel for sixty-five miles per hour for just one second unless there is a method for calculating the distance traveled. This calculational approach causes Tiffany to misunderstand another question from the same session (Table 3).

[Table 3 about here]

In line 2 of Table 3, Tiffany assimilates the question to a partitioning scheme, and answers in terms of the calculations she expected she would have to do, thus correctly answering a different question than the one intended. Tiffany's difficulties with speed were not just problems with partitioning --- in fact, she partitions easily here. Her problems with speed were also stemmed from the calculational approach she took to the problem. At least for a time (Lines 1 & 2), her understanding of speed as a mathematical object was disconnected from her experience of speed as a driver. Tiffany did not seem capable of imagining ‘any’ and ‘all’ partitions because she had to actually attempt to partition 45 mph into so many miles per minute before she could answer a question about the first minute.

Derek's partitioning

Derek, Tiffany's classmate, was also a high performing student in his non-honors Algebra II class, but unlike Tiffany, Derek had a fully formed fraction scheme capable of imagining "any" and "all" partitions. Although a full treatment of Derek requires more space than is reasonable here (see Castillo-Garsow (2010) for a more detailed description), the following example of the two students may help illustrate the differences in their partitioning schemes: contrast Tiffany (Table 4) and Derek's (Table 5) responses to the problem of using linear interpolation to estimate the value of a quarterly compounded account after point six years. The students were in separate sessions and did not hear each other's responses.

[Table 4 about here]

[Table 5 about here]

In Table 4, Tiffany was working with units of "quarters" that she was counting as integers using two levels of units, however she was not able to coordinate these quarters with the remaining tenth of a year, since it required an additional level of units not part of her counting quarters behavior. When presented with a number that wasn't a whole number of quarters, Tiffany's reaction was that she needed to find how many units (how many quarters) the number was in order to figure out how much a year it was, indicating that she was no longer using year as a unit to which she could reference the remaining tenth.

In contrast, Derek (Table 5) partitioned .6 years into .5 years and .1 years (used later). He then converted the 0.5 years to two quarters, without losing track of the original unit of years (three levels of units), and used the exponent “two” in the standard compound interest equation. The .1 he kept in units of years and put in the linear interpolation function that "works with fractions of a year under a quarter."

Derek's understanding of fraction partitioning came in part from an understanding of change that necessarily entailed continuous variation between quarters. As seen in lines 1 & 5 of Table 6, Derek used hybrid smooth variation in his understanding of the compound interest situation. He imagined that time passed continuously as he was thinking about the problem passing from zero through all the values in between on its way to a quarter of the year and that the account value in dollars was similarly growing in experiential time (as well as conceptual time) through all the values between $500 and the value at the end of the quarter.

[Table 6 about here]

My claim is that this image of hybrid smooth variation ("using smooth to consider chunks") facilitated Derek's thinking at three levels of units and allowed him to easily engage in partitioning .6 years into two quarters and remaining .1 years, while Tiffany who was engaged in chunky variation with chunks of a quarter could not accommodate to .6 years. Furthermore, I claim that the smooth component of Derek's hybrid variation is a way of thinking that he could separate from chunky variation and that pure smooth variation was something Derek used to make mathematical conclusions on its own, as we will see in the next section.

Derek's smooth variation

The examples from this section comes from Derek and Tiffany's work with the phase plane, specifically from the problem of using the graph of continuous compounding in the phase plane (Figure 4) to construct a graph of the value of the account over time from an initial investment of $500. The two students worked this problem in separate teaching episodes, with Derek's episode occurring chronologically first.

Derek solved the problem immediately, saying "as long as your the money in your account is growing then so will the rate of growth will grow so then it will just keep going up," and when asked to draw a graph of the first two seconds of the account, he drew the graph in Figure 5. Although Derek's solution shows chunking at one and two seconds, the verbal explanation that Derek gave while generating the graph was pure smooth variation with no indication of a unit. Both are the hybrid and the pure smooth approaches generate correct solutions to the problem.

[Figure 5 about here]

In asking Derek to elaborate on his solution, the qualitative (pre-numerical) nature of the reasoning becomes more apparent in Table 7. Derek imagined that as time was passing, the amount of money in his account was increasing. Since the rate of growth of the account was tied to the amount of money in his account, Derek imagined the rate of growth of the account was also increasing, so the account was growing "faster and faster." As the account grew faster and faster, Derek imagined that the rate of change --- still tied to the account --- was also growing faster and faster. All of this occurred in the present time. Derek was imagining the account time passing as he was giving his explanation, and he was tracking time, account value, and account rate as time was passing, in the present tense, for both him and the account. This smooth, qualitative image of an account growing faster and faster as time passed both conceptually and experientially is what informed Derek's sketch in Figure 5.

[Table 7 about here]

Tiffany and the phase plane

Despite appearances, the method that Derek used to solve the phase plane problem is not simple or easy. The sophistication of Derek's solution and the critical importance of imagining *experiential* change in progress to such a solution can be seen in Tiffany's work on the same problem.

Tiffany's work with the phase plane followed Derek's and I attempted to teach her Derek's method of solving the phase plane by asking her to use the previously learned fingertool method of coordinating changing quantities on a graph (Lima et al., 2009; Thompson, 2002) However, Tiffany was having difficulty coordinating three quantities at once (the dollar amount, the year, and the dollar per year rate), so we began by coordinating finger positions over regular time increments (Table 8).

[Table 8 about here]

As the session progressed, I pushed Tiffany to think in smaller and smaller time increments and become more and more qualitative (less reliant on explicit numbers) in her judgments (Table 9), with the goal of her eventually moving and thinking continuously[5] (Table 10).

[Table 9 about here]

[Table 10 about here]

Immediately after Tiffany described a "single steady movement" in line 4 of Table 10, I asked her to sketch a graph of the first three second of the account. Rather

[5] This turned out to be much more difficult than anticipated (see Castillo-Garsow (2010) for a listing of the few times Tiffany successfully thought continuously), and finding successful ways to cause students thinking in chunks to instead imagine smooth change is an area of research that I am actively engaged in.

than continue with the thousandths of a second single steady movement reasoning, Tiffany interpreted the problem as being about three iterated one second chunks saying, "Start with there, and then after that first three second. Oooh. Three seconds. We're not thinking of hundredths of seconds. That's gonna be like err after one second." She drew the graph in Figure 6.

[Figure 6 about here]

When asked to think in smaller increments, Tiffany repeated the unit-by-unit reasoning described in Table 9 drawing additional points based only on the local relationship between that point and the previous point giving no overall sense of what the graph might look like if the all the points were filled in (Figure 7). She described the overall function as "jagged" (Table 11).

[Table 11 about here]

Tiffany's solution to the phase plane problem (Figure 7, Table 11) shows both the mathematical and conceptual limitations of partitioning. Mathematically, no matter how small something is partitioned, the partitioning (the rational numbers) can never fill in the entirety of the real line. Conceptually, the act of partitioning entails that there be something between the partitions that is not being partitioned. Tiffany was *always* aware of this stuff between the partitions and represented it as holes in her graph (Figure 6, Figure 7). Even in the case where Tiffany described a "smooth steady movement" Tiffany was still aware of the jumpiness of the movement (Table 10, line 4). She said the jumpiness would be too small to see, implying that she was still aware of its presence. And so I claim that no amount of partitioning can ever achieve "all," but rather a mixture of partitioning (chunky reasoning) supported by smooth variation is necessary to achieve an image of "all." A recursive distributive partitioning scheme that supports *any but no particular* partitioning *may* rely on a coordination between distributive partitioning and experiential change in progress, but a conception of intensive quantity that incorporates "a variable state that symbolizes *all* comparisons" certainly requires smooth variation.

Chunky and smooth proportion

Another (brief) example of the distinction between partitioning and reasoning about "any" or "all" cases comes from the following excerpt from a high school Algebra I teacher (Table 12).

[Table 12 about here]

Steffe et al. comment that "But no matter how many particular partitions are chosen, collectively they are not sufficient to establish the concept of density... so that any but no particular partition can be conceptualized to establish ratios equivalent to one gram of water per cubic centimeter" (pp. 18, this volume). In the Augusta example (Table 12), the teacher is hitting a similar problem of trying to teach an concept of rate for "any" change in x from examples of particular partitions. But this is impossible. Augusta describes acts of partitioning one lines 1 & 2, but these lines do not establish the value of the ratio necessary to answer the question about "any" in the way that she wanted it to be answered. It is not until she gives the value of the ratio on line 3 that the students are able to respond.

The two different meanings of proportion (partitioning and constant multiple) that Augusta juggles in this example are illustrative of the distinction between two ways of imagining proportion that I call *chunky proportion and smooth proportion*. Chunky proportion is based on coordinating partitioning of the variables that make up the proportion, as Augusta does on lines 1 & 2. It establishes equivalent ratios by using tools that function for *any* proportion but do not identify the proportion itself (the value of the ratio). In the case of Augusta's question, a chunky proportion is imagining that whenever the change in x is partitioned into some number of pieces (ex: a third of the change in x), the change in y is partitioned into the same number of pieces as the change in x (ex: a third of the change in y), true of all proportions.

In contrast, a smooth proportion imagines the variables changing continuously in experiential time through all possible values and looks for a relationship unique to that proportion: the functional constant multiple from one variable to the other, or constant ratio between changing variables. In the case of Augusta's question, a smooth proportion would be imagining that as the change in x (itself a variable) changes through all possible values, the variable change in y is always -3.1 times as large as the variable change in x, true of this particular proportion.

Discussion

The case of Tiffany, the high performing Algebra II student with only chunky thinking shows that neither smooth thinking nor advanced chunky thinking (involving three levels of units) is necessary for success in high school mathematics (as defined by measures such as grades on tests). But success in high school mathematics is not the same thing as an understanding suitable for undergraduate mathematics used in the sciences and engineering (calculus and differential equations).

The case of Derek who had both smooth thinking and advanced chunky thinking shows the importance of both of these ways of reasoning in higher mathematics and in school mathematics. In particular, Derek's smooth reasoning enabled him to work with both undergraduate topics such as differential equations and high school topics such as speed and the partitioning of .6 more quickly and more easily than Tiffany. If leaving the door open for students to take and do well in undergraduate level math and sciences classes is a concern, then more research is needed in how to a) help students see and make use of opportunities for smooth thinking, and b) prevent students from becoming like Tiffany and losing track of the role of sensorimotor experience in mathematics in the first place.

Also note that what Tiffany was missing was not a sophisticated or advanced idea such as Thompson's smooth variation over recursive partitions, but rather the basics of pure smooth reasoning applied to mathematics. This is particularly notable in the speed discussion, where Tiffany, a student who drives, does not recognize the need to reflect on her own sensorimotor experience of driving as a way to answer a mathematics question. Although Tiffany was missing the third level of units, simply operating at two levels of units does not explain her difficulties with the speed questions. What students need is tools of both chunky (counting, iterating, partitioning) and smooth reasoning (including reflecting on sensorimotor experience) coordinating and supporting each other in the development of more advanced schemes (such as Thompson's recursive hybrid variation).

The case of Augusta shows the importance of treating smooth thinking and advanced chunky thinking separately, and not assuming that a student having one necessarily entails the student having the other.

Steffe et al. cite Thompson's definition of quantity as " as a scheme consisting of an object concept, a property of that object concept, and an appropriate unit with which to measure that property" (Thompson (1994) as cited by Steffe et al. this volume). They further quote Thompson in a personal communication describing the limitations of this definition of quantity in that it "does not make explicit that object concepts and their properties are constructed, that those constructions are most often nontrivial and highly problematic for children, and that the idea of measure is at heart proportional" (Thompson (personal communication) as cited by Steffe et al.).

I interpret this second quote as emphasizing the importance of images (Piaget, 1967; Thompson & Thompson, 1992) in quantitative reasoning: that developing a quantity requires developing an image of the object and its property. In the case of intensive quantities especially, successful construction of the quantity requires building a smooth, dynamic (experiential time) image of the object and its property prior to quantification. For example, Steffe et al. note that the decision to establish the ratio as a definition of density on "an intuition of density." (p. 18) I claim that developing an "intuition of density" depends upon the development of this type of smooth reasoning.

What we have in Steffe et al.'s paper is a meticulously detailed account of how *chunky* schemes of fraction, ratio, rate, and intensive quantity develop. What we need to fill in the gaps that Steffe et al. identify and to account for the behavior of students like Derek is a similarly detailed account of how *smooth* schemes (schemes that might fall in the smooth category) of the same topics develop and contribute to Steffe's chunky schemes.

There are already some candidates for ideas that may be developed though future research into schemes of operations that describes types of smooth reasoning. With further development, smooth proportion may become a theoretical tool with similar explanatory power to Steffe et al.'s schemes of fraction and rate. Steffe et al.'s definition of *extensive quantitative variable* as "the potential result of measuring a varying but unknown extensive quantity at any but no particular time" (p. 4, this volume) may also either be a smooth scheme or rely on a smooth scheme, particularly in imagining a "varying [present tense] but unknown extensive quantity" prior to imagining the potential result at any but no particular time. Additionally, in a personal communication, Steffe has suggested the possibility of three levels of smooth reasoning based on interiorization. I do not yet have the data to support this level of detail, but encourage Steffe to continue this line of thinking if he does.

Another candidate with potential to become a scheme is Johnson's notion of intensity of change (Johnson, 2012). Documented cases of smooth reasoning such as Derek (Castillo-Garsow, 2010, 2012), Hannah (Johnson, 2012), and Zac (Moore, 2010, 2012) have so far all used very similar language when describing rates that seem to have origins in describing smooth images in terms of intuitistic notions of speed (increasing faster, increasing slower) that Johnson calls intensity of change. I would very much like to see a further development of this idea that identifies the origins and consequences of this type of language as part of a sequence of smooth and hybrid schemes.

Johnson's ideas provide a good starting point, but there is no current work that explains and predicts the behavior of students like Derek, Hannah, or Zac to the level that Steffe et al.'s work in chunky schemes is capable of. Smooth reasoning is a currently untapped resource for schemes, and investigating students from this perspective has great potential for building epistemic algebra students that we would not otherwise create.

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Table 1

Excerpt from discussion of simple interest

1	TIFFANY	So if you do a quarter of a year, or you just take three 12ths — literally, a quarter of a year,
2	TIFFANY	then you should get the same thing.
3	TIFFANY	And then the one quarter we're just looking at a quarter of a year.
4	TIFFANY	We're not really looking specifically at months or days or anything.

Table 2

Excerpt from discussion of speed

1	PAT	If I'm going sixty-five miles per hour what does that mean?
2	TIFFANY	In one hour, you've gone, you should have gone sixty-five miles. (Approximately 3 minutes of instruction in partitioning omitted)
3	PAT	Can– can I travel for just one second at sixty-five miles per hour?
4	TIFFANY	No you have to do — you would have to do... um... Well yeah you could.

Table 3

Excerpt from discussion of speed

1	CARLOS	So let's imagine that I was driving a car, umm, at 45 miles an hour and I did that for fifteen minutes. When those fifteen minutes were up, umm, I speeded up to sixty-five miles an hour. How fast was I driving in the first minute?
2	TIFFANY	You would have to convert the 45 miles per hour into miles per minute. Sorry, or feet would it be ffff- no cause you want to know how many miles right you've gone in the first minute?
3	CARLOS	No I wanted to know how fast I was driving.
4	TIFFANY	Oh how fast. Oh [laugh] umm, well you're still driving forty five miles an hour in the first minute.

Table 4

Excerpt from discussion of compound interest

1	TIFFANY	Point six *years*?
2	TIFFANY	Uh kay..., well we would have to figure out how much of the year that really is.
3	TIFFANY	Well six tenth's of a year. Like we could figure out how many quarters that is or something and then so the same type of thing...
4	TIFFANY	Like if it's just point six, I don't know right now I don't know how much of a year oops that really is.

Table 5

Excerpt from discussion of compound interest

1	DEREK	You'd bring it down to point five.
2	CARLOS	OK, so I bring it down to point five, and what does that tell me?
3	DEREK	It gives you, uh whole number, that you can put into *n*
4	CARLOS	OK, which is...
5	DEREK	Two.
6	CARLOS	Two, cause point five, is half a year, it's two quarters.

Table 6

Excerpt from discussion of compound interest

1	DEREK	It'd be going by eight percent of five hundred until it gets a quarter of the year.
2	DEREK	Then, so it's getting eight percent of five ten for a quarter of the year.
3	DEREK	And then eight percent of that for a quarter of the year and eight percent of that for a quarter of the year.
4	CARLOS	And what is it doing in between?
5	DEREK	In between it's growing at the rate just for that quarter of a year.

Table 7

Excerpt from discussion of phase plane

1	CARLOS	You know the rate is growing and so can you show me how the money is your in your account is growing?
2	DEREK	Umm on that axis?
3	CARLOS	By moving your finger along this axis yeah
4	DEREK	Like starts slow and then just keeps getting faster and faster
5	CARLOS	OK, umm and what about the rate of growth?
6	DEREK	It would also start slow and keep getting faster and faster.

Table 8

Excerpt from discussion of phase plane. Tiffany is moving her fingers along the phase plane (dollar amount and dollar per year rate) while she instructs Carlos on how to move his fingers on the time graph (years and dollar amount)

1	TIFFANY	So, and now I am at like five hundred and ten I'm still earning the forty dollars.
2	CARLOS	OK
3	TIFFANY	So now, like, so five hundred and ten if I moved another quarter of the year
4	CARLOS	So I move a quarter of a year
5	TIFFANY	That would be five hundred and twenty dollars
6	CARLOS	So I move to five hundred and twenty dollars
7	TIFFANY	OK

Table 9

Excerpt from discussion of phase plane. Tiffany is moving her fingers along the phase plane (dollar amount and dollar per year rate) while she instructs Carlos on how to move his fingers on the time graph (years and dollar amount)

1	CARLOS	And then when a second goes by how much should I move my finger?
2	TIFFANY	A little bit higher than what you were and then a little bit higher
3	TIFFANY	So another second goes by then you're gonna be moving a little bit more than before.

Table 10

Excerpt from discussion of phase plane.

1	CARLOS	In thousandths of seconds could you see the jumpiness?
2	TIFFANY	I don’t think so it would just look like a movement.
3	CARLOS	OK.
4	TIFFANY	A single steady movement. Cause if you couldn’t see it cause that’s really small. [laugh].

Table 11

Excerpt from discussion of phase plane.

1	CARLOS	Now how would you fill in what's going on in between?
2	TIFFANY	I think like we're going take this use that and then we take that and we use it meaning all of that to get uh that and then we just all the way up to here.
3	CARLOS	OK, so what would it look like?
4	TIFFANY	Kind of, where it's always kinda like going up till you get to the end kinda thing umm like kinda jagged kinda like.

Table 12

Excerpt from discussion of point slope form of a line.

1	AUGUSTA	How much should y change for each change in x? Well we said in half the time we would expect half the distance. If I triple my time frame I would expect to triple my distance. Kay?
2	AUGUSTA	Well however much I change my x value, whether it is by half or by triple three, however much I change that x value how would you expect your y value to change?
3	AUGUSTA	If the rate of change is negative 3.1
4	STUDENTS	Negative 3.1
5	AUGUSTA	Yes! Negative 3.1 times as much.

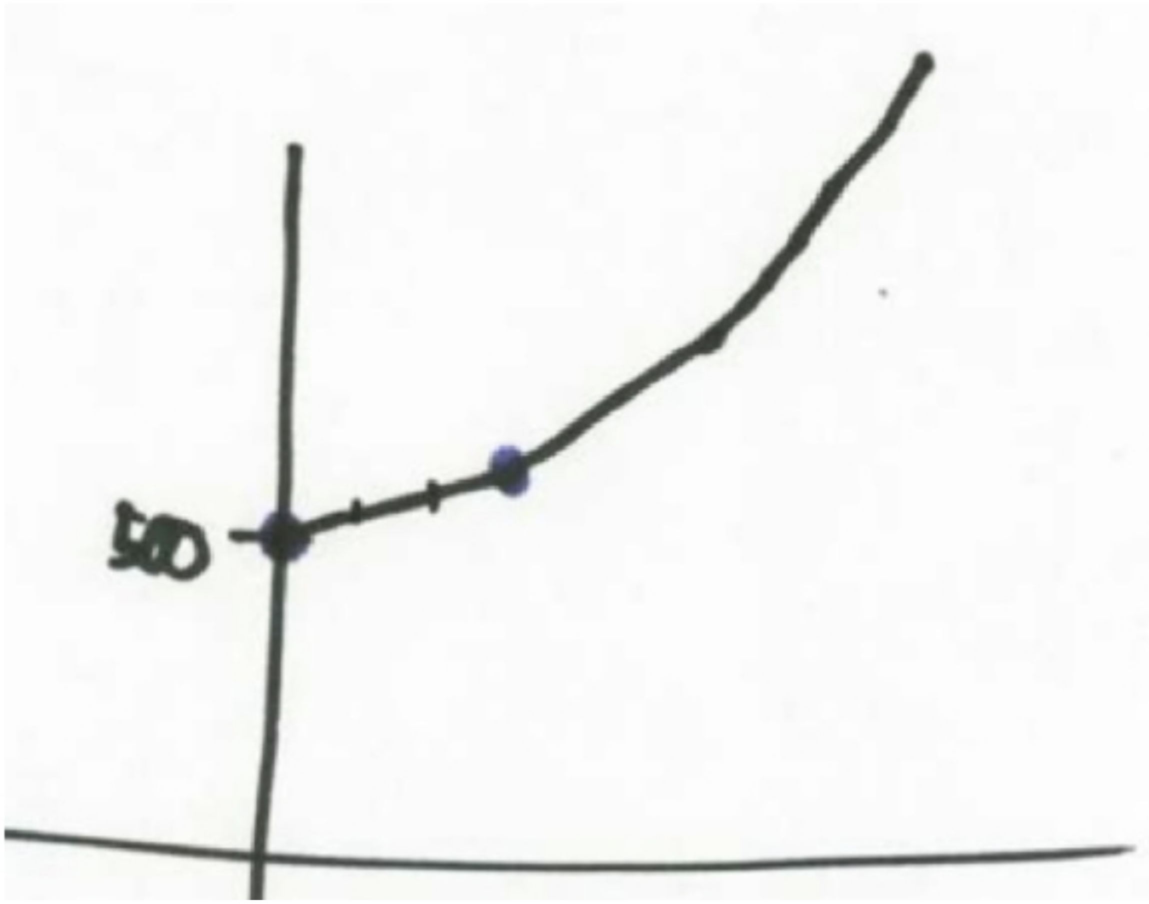

Figure 1 . Derek's graph of the compound interest function: linear growth where every quarter of a year (horizontal axis) the investment grows at a new rate and the graph changes slope.

$$500\left(1+\frac{.08}{4}\right)^0 + .08(500)x$$

$$500\left(1+\frac{.08}{4}\right)^1 + .08(510)\left(x-\frac{1}{4}\right)$$

$$500\left(1+\frac{.08}{4}\right)^2 + .08(520.2)\left(x-\frac{1}{2}\right)$$

$$500\left(1+\frac{.08}{4}\right)^5 + .08\left(500\left(1+\frac{.08}{4}\right)^5\right)\left(x-\frac{5}{4}\right)$$

$$\frac{5}{4} < x < \frac{6}{4}$$

Figure 2 . Tiffany's equations for the compound interest account showing both the piecewise linearity and the traditional geometric compound interest formula that emerges. Her bounds for each piece of the function were written elsewhere. Derek's equations were similar.

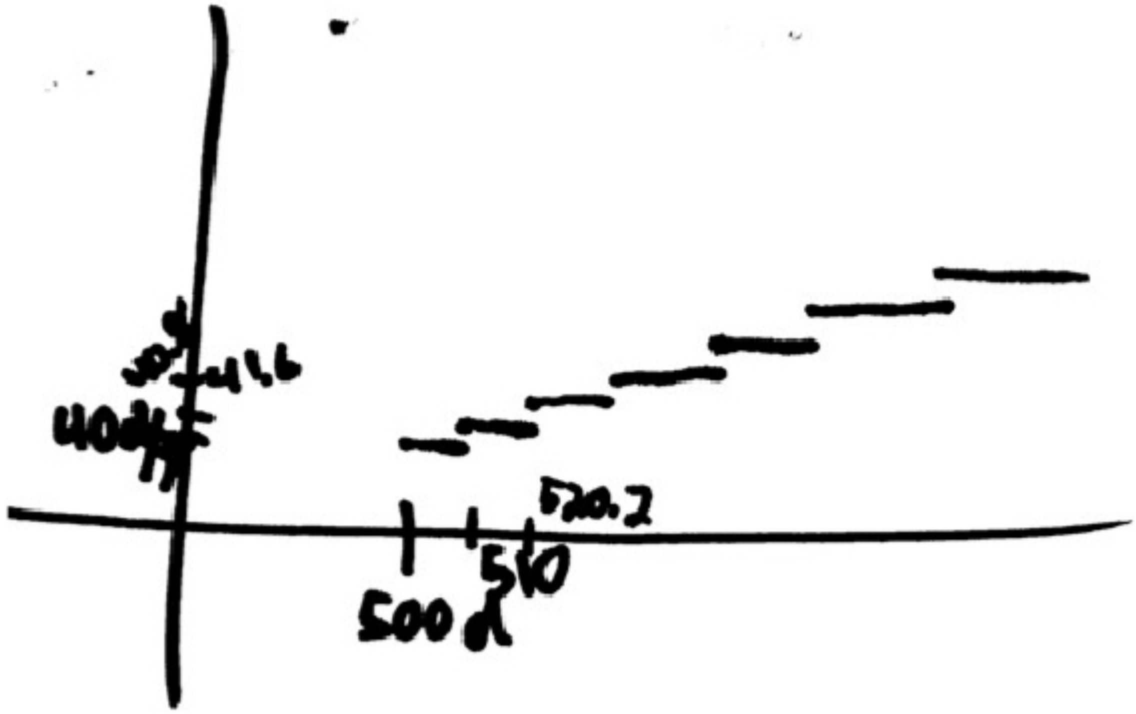

Figure 3 . Derek's graph of compound interest in the phase plane, showing that the rate of growth is constant over each compounding interval and that the account earns more money in each compounding interval.

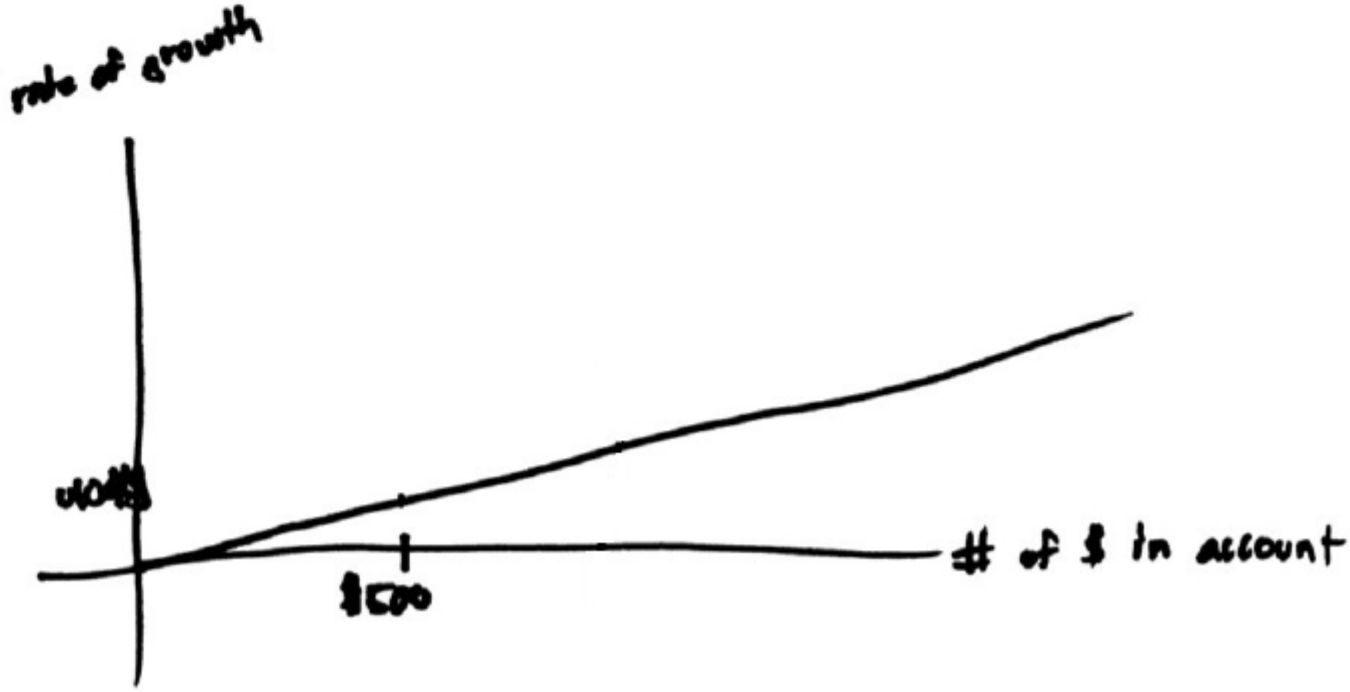

Figure 4 . Derek’s graph of continuous compounding interest in the phase plane.

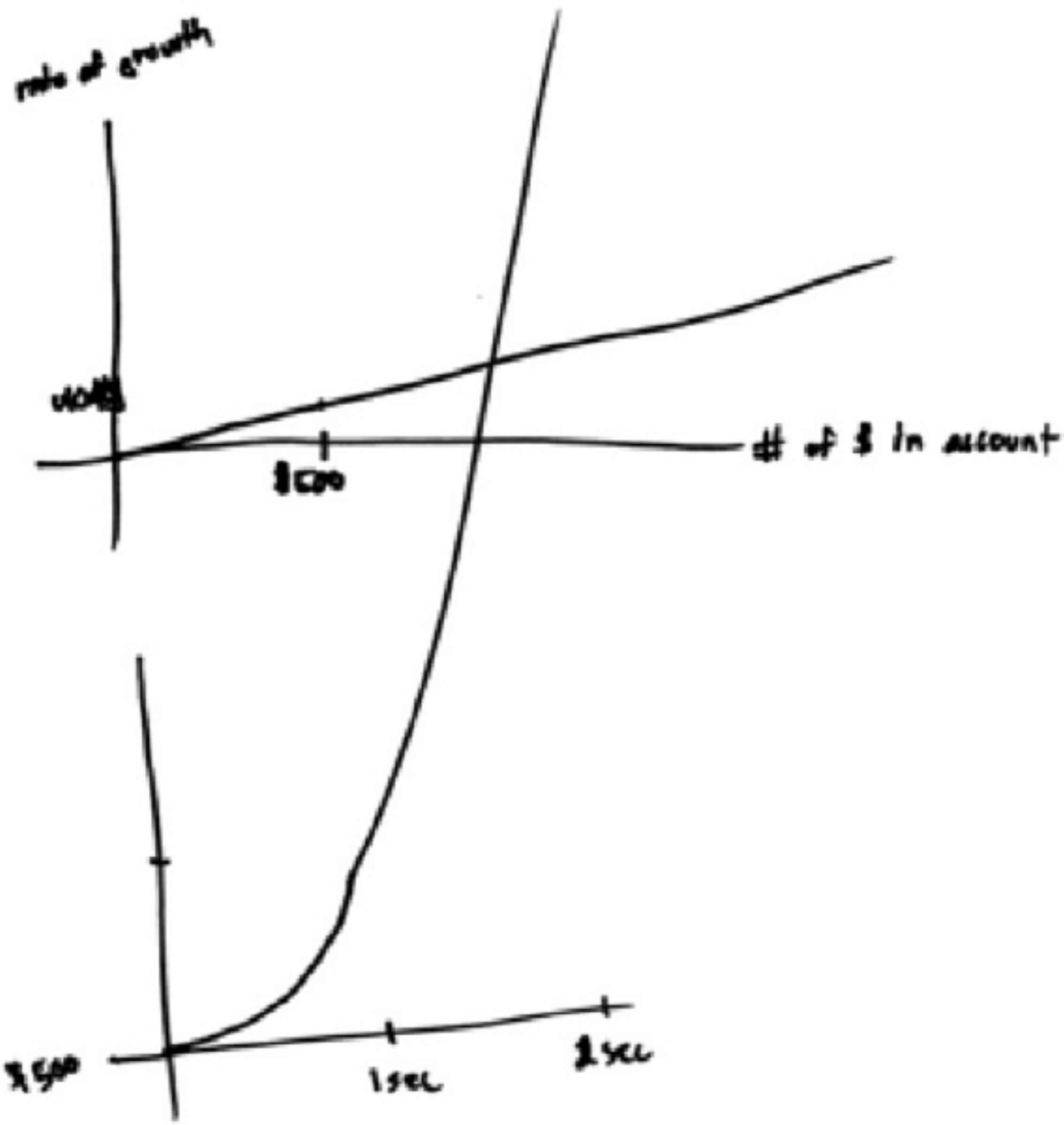

Figure 5 . Derek’s solution to the phase plane problem, drawn in two motions.

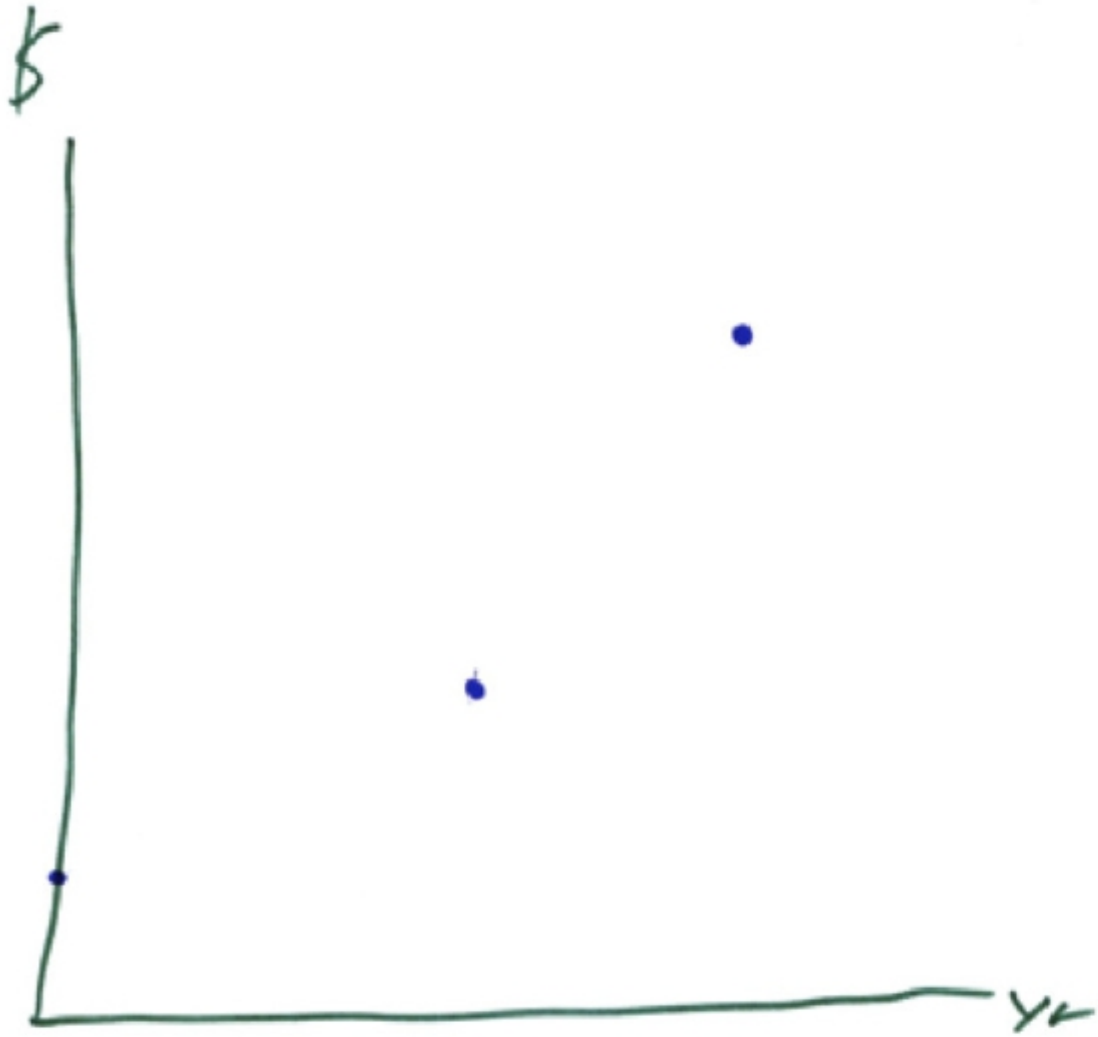

Figure 6 . Tiffany's solution to the phase plane problem, drawn over the first three seconds of the account.

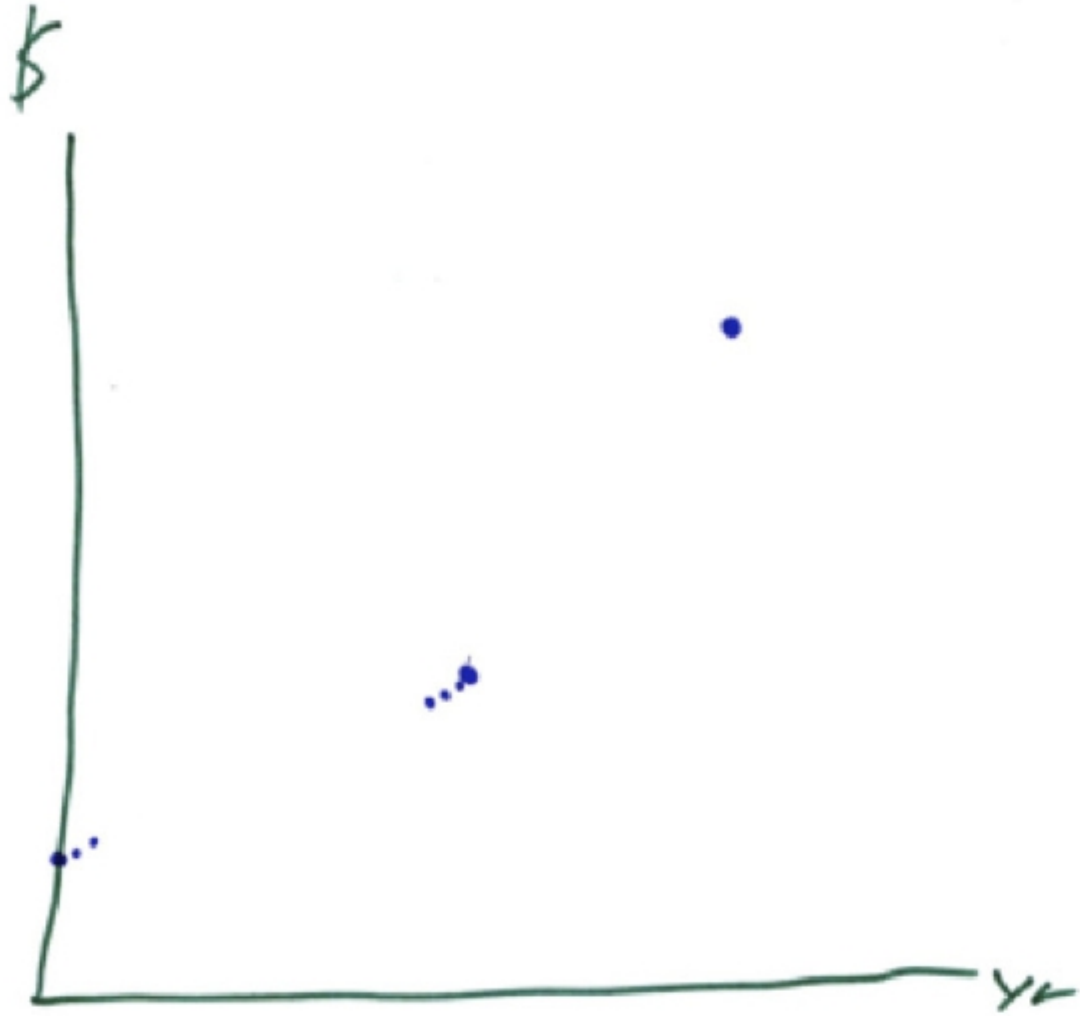

Figure 7 . Tiffany fills in one second point by point using local information only.