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THE ROLE OF MULTIPLE MODELING PERSPECTIVES IN STUDENTS' LEARNING OF EXPONENTIAL GROWTH

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Abstract. The exponential is among the most important family functions in mathematics; the foundation for the solution of linear differential equations, linear difference equations, and stochastic processes. However there is little research and superficial agreement on how the concepts of exponential growth are learned and/or should be taught initially. In order to investigate these issues, I preformed a teaching experiment with two high school students, which focused on building understandings of exponential growth leading up to the (nonlinear) logistic differential equation model. In this paper, I highlight some of the ways of thinking used by participants in this teaching experiment. From these results I discuss how mathematicians using exponential growth routinely make use of multiple — sometimes contradictory — ways of thinking, as well as the danger that these multiple ways of thinking are not being made distinct to students.

1. **Introduction.** The purpose of this manuscript is to highlight and generate a dialogue within the applied mathematics community interested in mathematics education, around important but problematic questions like "What do we really mean when we say we want high school or early undergraduate students to understand exponential functions?" The perspective that I bring into these questions has been shaped by my training in mathematical biology and mathematics education. Yet, it should be clear that this discussion would be enriched by those involved in research in other branches of mathematics or mathematics and science education including statistics and physics.

At a foundational level, it must be observed that exponential (and geometric) growth is a particularly important topic to *mathematics literacy* [1], and *mathematical biology education* in particular. The dynamical systems used to model populations [2]

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[1]In the news, the importance of exponential literacy can been seen in part in the success of sub-prime mortgages the United States and the resultant worldwide financial crisis

[2]The use of geometric growth to model populations can be traced back at least as far as Thomas Malthus [10]

take as their basis the exponential function (via linearization and the study of local dynamics) with the outcome of exponential growth being particularly relevant in the context, for example, of biological invasions[3]. The exponential (geometric) function is also the core building block, via superposition, of the solutions to linear differential (difference) equation models. The fact that exponential growth receives particular attention in this study is amply justified by its importance in these systems.

Exponential growth, typically taught in high school algebra and re-taught in undergraduate level mathematics and biology courses, can be thought of (and taught) in any number of ways. Although we are seldom thinking about it, mathematical researchers and educators are intimately familiar with several ways of thinking about exponential growth. Hence, it is not surprising to see professional mathematicians bring simultaneously multiple ways of thinking/understanding of exponential growth, that are unconsciously managed in real time. Professional mathematicians and educators switch without blinking between thinking of exponential (and geometric) growth in ways that include its view as the result of an iterative multiplication process; or the outcome of a relationship between a population of individuals and their collective growth contributions; or as the outcome of a stochastic process. We do it as easily as a multilingual speaker switches between languages in mid-sentence. Just as a the bilingual speaker changes languages to choose the words that best convey their meaning, as mathematical biologists, we instinctively change our exponential growth perspective to best suit the needs of the problem in the moment. We use multiple ways of thinking about exponential growth concurrently and interchangeably, taking for granted (unconsciously) that these different ways of thinking not only follow easily from each other but that our audience is capable of connecting the dots, in real time.

Contrast the above perspective with that of a student who is just learning about "exponential growth" in middle or high school for the first time. They know from the use of the word "exponential" in casual conversation or television that it means something like "a really big change," "grows really fast," or "curves up like this" none of which highlight the mathematical properties of the exponential function that mathematicians and biologists rely on. From this starting point, students are typically introduced to exponential growth via situations such that include grains of wheat on a chessboard or bacterial growth where they are required to multiply repeatedly. They then begin to build geometric sequences and use of classroom introduced exponential notation to describe these situations. The fact is that there are huge jumps between this casual, non-mathematical meaning of "exponential" based in community use (common language), the "memorization" of examples or representations associated with geometric growth, and the abstract appropriation of the concept of exponential growth based on the mathematical properties of the function. These gaps are either unconsciously ignored or assumed to be only within the reach of those who have mathematical talent.

The question "What do we really mean when we say we want high school or early undergraduate students to 'understand exponential functions?'" is the driver of this article. In order to pose the centrality of this question we address it in two parts. First I highlight some of the disadvantages of the most common approach of teaching students exponential growth via repeated multiplication and the geometric sequence. Secondly, I highlight the results of a teaching experiment that

[3]a process that not only drives so much research that even has its own journal — *Biological Invasions*, published by Springer

shows some weaknesses of another common approach — compound interest — as well as that high school students are capable of developing understandings of exponential growth far more sophisticated that what is typically taught in high school, using only a small mathematical toolkit. If these sophisticated understandings are mathematically accessible to high school students, then we have a greater number of options in terms of what to teach as exponential growth, and this dialogue becomes important. A dialogue started from the suggestion that mathematicians, mathematical modelers, and mathematics education researchers — all of whom have teaching responsibilities — together must systematically consider a broader number of options when introducing exponential growth, and that a single approach (way of thinking) to teaching exponential growth may not be enough.

2. **Filling in the gaps is not trivial.** A prototypic approach to teaching exponential growth in a high school class is to introduce a problem that requires repeated multiplication to solve, and then connect that repeated multiplication to exponential notation so that the student writes an exponential function. However there is a significant problem of how to extend that geometric sequence to exponential growth in the real numbers. In many cases, students are simply expected to connect the dots with a curve because that is what the pattern of dots looks like, or because that is what the function they found graphs like on a calculator. If memory serves me correctly, my own introduction to this problem was simply from the connect-the-dots perspective, without any sort of mathematical or modeling justification, and this did not bother me at the time.

This problem is approached differently in modern mathematics education literature [5, 6, 14]. What Confrey & Smith imply, and Strom states explicitly is a mathematical process of filling in the gaps based on the composition of operations. If one imagines a geometric sequence S_n generated by the repeated multiplication of growth factor 3, then a fraction term in the sequence such as $S_{\frac{1}{4}}$ could be generated by imagining an operation that when composed 4 times results in an operation of multiplying by 3. In this example, the operation would be multiplying by $\sqrt[4]{3}$. This process can be extended to any growth factor n and any fraction $\frac{p}{q}$, resulting in an extension of any geometric sequence to the rational domain. Strom refers to the process as "partial factors."

However, in the context of mathematical biology or other modeling disciplines, the partial factors solution is at its core a mathematical approach, not a modeling approach. It relies on the properties of rational exponents to generate the curve, rather than a biological or physical mechanism. In fact, due to non-sequential nature of the "filling in," it seems unlikely that there could exist any time based physical or biological process that would generate this mechanism of deriving the population at fractional values of time from the population at whole values of time.

Furthermore, there is a question of whether or not we want early *modeling* students to fill in the value of a geometric sequence with an exponential curve at all. Consider the following situation, which is not unreasonable for a student first learning to model exponential growth:

> A single bacterium splits into two bacteria every five minutes. How many bacteria will there be after t minutes?

This is the sort of situation that is initially well suited to a geometric sequence (Figure 1). A student might imagine that because every five minutes each bacterium doubles, as a whole the population also doubles every five minutes, resulting in

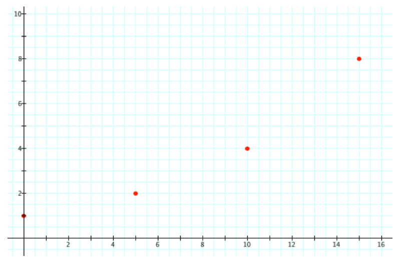

FIGURE 1. Modeling bacteria growth with a geometric sequence.

repeated multiplication by two, resulting in the equation $f(t) = 2^{t/5}$. However, this repeated multiplication by two begins to break down when questions are asked about the value of population after one minute, or after 0.783491. minutes. For example, our function f predicts a value of 1.1487 bacterial cells at one minute, which is biologically impossible. More importantly, the gradual growth of cells through all real values does not fit the likely biological image of the student.

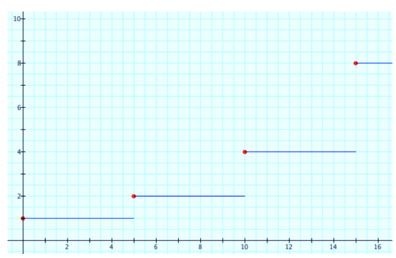

FIGURE 2. An extension of geometric growth in real valued time with non-overlapping generations.

A reasonable interpretation of this situation is that the cell waits for five minutes, then replicates, and that the subsequent cells wait for five minutes without changing, and then replicate all at once, and so on (Figure 2) (non-overlapping generations). In a situation like this, the value of the population at one minute would be one cell, because the cell has not yet replicated, and the function for this interpretation would be $g(t) = 2^{\lfloor t/5 \rfloor}$ (Figure 2).

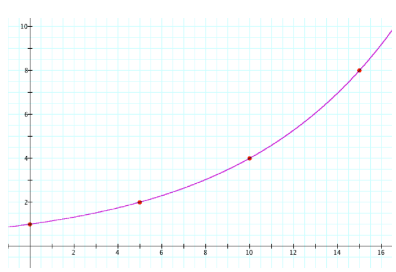

FIGURE 3. Modeling geometric bacterial growth with an exponential function results in different values than Figure 2.

There exist other interpretations of this situation that would result in an exponential curve (Figure 3). However these interpretations require much more sophisticated biological and mathematical thinking. Examples include: The time to division is exponentially distributed with parameter 5, and the function reports expected population; the population is being measured continuously as biomass, and we assume that the rate of growth of the biomass is proportional to the current biomass; or the population is extremely large with asynchronous overlapping generations, and the function returns an estimate for which fractional values can be safely ignored as inconsequential. These are quite sophisticated perspectives that would confuse most individuals seeing this concept for the first time.

All of these interpretations are the result of sound modeling practice, but they do not reflect the understanding of the student first learning exponential growth with the aid of modeling. This presents us with a two-fold dilemma. If one has the goal of teaching good modeling then the step function model is the best modeling practice — it is the most accurate representation of the student's understanding of the situation. However, if one has the goal of teaching continuous exponential reasoning, then one must either gloss over the student's non-exponential understanding of the situation, change or abandon the context entirely, address these modeling complexities, or teach them something more sophisticated than the geometric sequence model. The first alternative is not acceptable. The second may function as a ways of building continuous exponential growth in some other context, but this does not mean that geometric growth alone is sufficient. Since geometric growth alone cannot cover biological processes (and other contexts) other meanings must also be taught in order for students to be able to deal with these situations. What I will show below is that the last two options are viable as alternatives to geometric growth or as part of a multiple perspective approach that might include geometric growth as one perspective among many.

3. **Teaching Experiment.** In an effort to do the latter, I designed a teaching experiment [13] targeting the "rate proportional to amount" option from the list above. The full details of the teaching experiment can be found in my dissertation [3], however, I'll summarize the design for the purposes of the following discussion.

Over fifteen 50-minute sessions, I interviewed two high-school Algebra II students via a series of tasks designed for teaching them the Verhulst model (in the form of the logistic differential equation). The design of the tasks was based partially on Thompson's conceptual analysis of the exponential [15], and partially on the approaches used in mathematical ecology texts [2, 1]. The tasks themselves were primarily from financial modeling, but one student, "Derek," also worked with biological modeling, and I believe the results are far more significant to the learning and teaching of biological modeling than financial.

From Thompson [15], the design of the tasks began with constant rate of change and simple interest, establishing a function family where the rate of change (in dollars per year) of each line was proportional to the y-intercept of that line. From this simple interest line, students were asked to consider compound interest situations, where the bank would periodically update the (dollar per year) rate of change to be proportional to the value of the account at the beginning of each compounding interval, resulting in a piecewise linear compound interest function (Figure 4).

The remainder of the teaching experiment was built on the idea of "what if rate was proportional to the value of the account all the time?" And introducing some

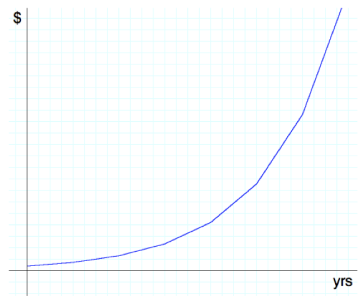

FIGURE 4. Beginning with simple interest, the bank updates the growth rate in \$/yr of the bank account at the beginning of each compounding interval, resulting in a function that is linear over intervals of uniform size.

well-disguised ordinary differential equations, of the type found in mathematical ecology [2, 1]. First the idea was introduced with "rate" as a simple function of the value of the account, and graphing this relationship in the phase plane without explicitly tying the value of the rate to the behavior of the account. The original design anticipated that the students would use their experience with compound interest to estimate the account behavior using a qualitative form of Euler's method, followed by the use the same methods for the logistic model in the phase plane. The methods that one student used, however, were very different from what I originally anticipated, and these methods will be the focus of discussion.

Over the course of the teaching experiment, the students used several different "ways of thinking" about exponential growth, some resulting in incompatible mathematical results. Derek in particular would use multiple ways of thinking during the same task. Rather than hinder him, these rapid switches between perspectives actually improved his understanding.

4. **Geometric Growth.** Although the design of the experiment was intended to avoid the geometric sequence as a way of thinking about exponential growth, the geometric sequence was also the way of thinking that the students were already familiar with. This familiarity showed up in what was – for me at the time – an unexpected way.

In the first teaching interview, I began by introducing the simple interest task, the text of which is reproduced verbatim here:

> Jodan bank uses a simple interest policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of eight percent of the initial investment per year. Create a function that gives the value of an EZ8 account at any time.

The intended interpretation of this task was that the students would imagine that the dollar per year rate of growth of the account would be .08 times the value of the account at zero years. Allowing for flexibility in the notation, I was looking for a function of the form $y = .08bx + b$, a solution which the students did eventually reach (Figure 5). However, initially, the students did not interpret the task in this way.

I began this task by asking Derek to read the problem aloud and then I asked Tiffany to explain the situation. During her explanation, Tiffany gave the following example:

> *Excerpt 1 – Episode 1, 00:02:29*
> 1 Tiffany: Like if you put in a certain amount – I would say like ten dollars beginning
> 2 Carlos: OK
> 3 Tiffany: And then that the next year it should have grown, like I don't really know what eight percent of that is, but
> 4 Carlos: Eighty cents
> 5 Tiffany: Thank you. Like should have – So now in the next year she'd be like ten eighty cause there's like from-
> 6 Carlos: OK
> 7 Tiffany: Eight percent more and then the next year it's like, you know, whatever eight percent of ten-eighty is, so it should be doing something like that. That's how I look at it that's how I see it.
> 8 Carlos: [To Derek] What do you think?
> 9 Derek: Yes.

In this excerpt's line 7, Tiffany describes taking 8% of the previous year's account value, which would generate a geometric sequence, rather than taking 8% of the initial investment, which would have generated an arithmetic sequence, or imagining that for any change in time, the change in account value would be .08 times the initial value, times the change in time, which would have generated a line.

Derek's "yes" on line 9 is used to indicate that he agrees with Tiffany's explanation. It was only when I asked the students to explain "initial investment" that they began thinking in ways that would lead them to linear growth (Figure 5).

$$q(x,n) = n + .08(n)(x)$$

Figure 5. Tiffany's final solution to the simple interest task.

This example is used to illustrate that while the geometric sequence is an idea that students are comfortable with, their use of the geometric sequence does not necessarily reflect a strong or carefully developed understanding of the problem situation. In this case, a lack of attention to the nuances of the phrase "initial investment" combined with the presence of the keyword "percent" that usually signals geometric growth seems like a reasonable explanation for this phenomenon.

5. **Piecewise linear compound interest.** Following the simple interest task, the students were presented with a new task, describing a modification to the simple interest policy.

> The competing Yoi Trust has introduced a modification to Jodan's EZ8, which they call the YR8 account. Like the EZ8 account, the YR8 earns 8% of the initial investment per year. However, four times a year, Yoi Trust recalculates the "initial investment" of the YR8 account to include all the interest that the customer has earned up to that point.

This situation introduces the idea of compound interest independently of the context of geometric sequence by basing compound interest entirely on the idea of linear rate of change. This situation did not have a clearly defined task for the students, but rather I asked the students questions about the situation that led them towards developing a function for a YR8 account with an initial investment of \$500. Tiffany's solution is provided below. Derek's solution was similar.

$$500 + 0.08\,(500)\,\frac{3}{12}$$

$$510 + 0.08\,(510)\,\frac{1}{4}$$

FIGURE 6. Tiffany's calculations for the value of a YR8 account after the first quarter (purple) and second quarter (green), showing their origin in simple interest reasoning (compare with Figure 5).

Tiffany began by calculating values at specific quarters, using what she had learned from simple interest. The pattern of these calculations (Figure 6) matched the format she developed in her simple interest formula (Figure 5). She developed them by reasoning that if after one year, the account would have grown by 500*.08=40 dollars, then after a quarter of a year, the account would have grown by a quarter of 40 dollars. This reasoning is consistent with the linear simple interest growth from which the YR8 account was derived.

I then set Tiffany to the task of rewriting her calculations in terms of the initial \$500. After some instruction in the distributive property, Tiffany was able to rewrite the formula for second quarter account value in a more traditional form (Figure 7).

$$\left(500 + 0.08\,(500)\,\frac{3}{12}\right) + 0.08\left(500 + 0.08\,(500)\,\frac{3}{12}\right)\frac{1}{4}$$

$$500\left(1 + \frac{0.08}{4}\right) + 0.08\left(500\left(1 + \frac{0.08}{4}\right)\right)\frac{1}{4}$$

$$500\left(1 + \frac{0.08}{4}\right)^2$$

FIGURE 7. Tiffany's substitution of the first quarter calculation into the second quarter calculation (cyan), followed by two factorings (yellow and gray) resulting in a traditional form for compound interest (gray).

After identifying the power of 1 in the first quarter calculation, and the square in the second quarter calculation, Tiffany hypothesized that then next quarter would involve a cube. In this way Tiffany generated a geometric sequence. However, it is critical to point out that this geometric sequence developed as result of the discussion rather than being a starting point for it. This difference can be seen in

Tiffany's final form of the function (Figure 8), which shows a much more sophisticated understanding of the linear origins of compound interest growth than the traditional compound interest formula.

$$500\left(1+\frac{.08}{4}\right)^0 + .08(500)x$$

$$500\left(1+\frac{.08}{4}\right)^1 + .08(510)\left(x-\frac{1}{4}\right)$$

$$500\left(1+\frac{.08}{4}\right)^2 + .08(520.2)\left(x-\frac{1}{2}\right)$$

$$500\left(1+\frac{.08}{4}\right)^5 + .08\left(500\left(1+\frac{.08}{4}\right)^5\right)\left(x-\frac{5}{4}\right)$$

FIGURE 8. Tiffany's function for a YR8 account with an initial value of \$500, showing both geometric and linear understandings. Tiffany did define the function on intervals, but most of these intervals were written elsewhere in her scratch work.

Working together, the students also described what a graph of this function might look like, and sketched it out (Figure 9), again showing an understanding of both the linear and geometric aspects of this function.

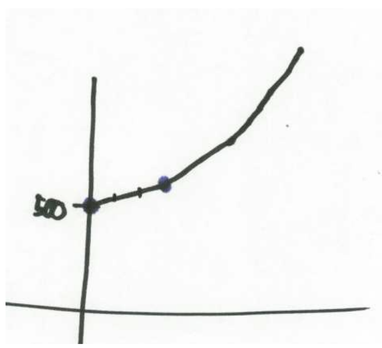

FIGURE 9. Derek's graph of the compound interest function, drawn in discussion with Tiffany. Cropped for detail.

6. **Euler's Method.** The next three "ways of thinking" about exponential growth all take place in the setting of a phase plane task (called the "PD8 account"). Specifically, the students were tasked individually with reconstructing an exponential function from its linear graph in the phase plane (Figure 10). Tiffany and Derek approached this problem very differently, with Tiffany calling upon her previous experience with compound interest, as I had originally intended, and Derek taking a variety of perspectives that were not part of the original design.

Tiffany's method was to imagine that at a certain time (0 years), the account had a certain value (\$500), and an associated rate of change (\$40/yr). She then imagined that as one second passed (a time interval I gave to her), there would be a certain amount of change in the account. This new value would lead to a new rate of change, so that as the next second passed, the change in the account value would

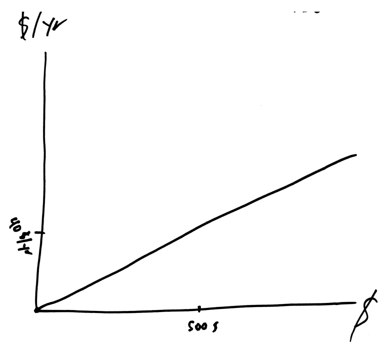

FIGURE 10. Tiffany's graph in the phase plane of a 'PD8' account. Her task was to use this graph to create a graph of the value of a PD8 account over time.

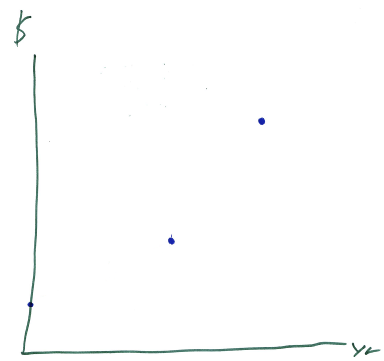

FIGURE 11. Tiffany's graph of the value of the PD8 account over time for the first two seconds.

be greater than it was before, and so on and so on. Using this method, Tiffany constructed a graph of an approximation to the PD8 account (Figure 11).

When I asked Tiffany how to fill in what goes on in between points, Tiffany repeated the process on a smaller scale, filling in point by point, and describing a process of using each point to find the next point. She described the overall function as "jagged."

In Figure 12, Tiffany filled in the first second of the account with two sequences of points, one sequence near 0 seconds, and one sequence near one second. Within each of the two sequences each change in height is higher than the one before, but the two sequences are not placed so that the sequences will connect, showing no sense of the overall shape of the curve. Tiffany was unable to describe overall shape of the curve verbally, either.

The key issue here is that although Tiffany was aware that her graph was an approximation of something, the compounding process she was using did not provide her sufficient information to identify what she was approximating. She placed points on what appears to us to be a curve, but only because we already know, by other methods, what the answer must be. To her, constructing the function point by point, without knowing the solution ahead of time was an extremely difficult task.

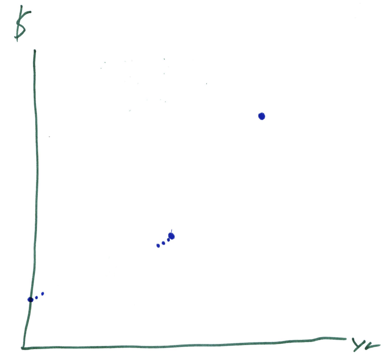

Figure 12. Tiffany fills in the first second of the account point by point.

Placed in this situation, the consistency in her reasoning is admirable. She did not abandon the compound interest methodology she had developed over the past two weeks in order to blindly "connect the dots." Unfortunately, her dedication to using only a single "way of thinking" rigorously inhibited her ability to imagine the approximation process in its limit. In the next two sections we will see how Derek introduced new "ways of thinking" to gain a better understanding of the exponential function, despite the inconsistency of his approach.

7. **Qualitative Differential Reasoning.** Derek was also given the task of reconstructing a graph of the account value over time from a phase plane graph, but unlike Tiffany, Derek's approach was not based on the compound interest reasoning used earlier in the teaching experiment. Instead, Derek answered the question about the behavior of the account immediately.

Excerpt 3 – Episode 11, 00:05:52
1 Derek: As long as your the money in your account is growing, then so will the rate of growth will grow. So then it will just keep going up.

Derek's answer in Excerpt 3 is remarkable in its brevity, but by asking him to place his finger on the x-axis to represent the value of the account, I was able to get him to slow down his explanation to the point where I could reconstruct his thinking.

Excerpt 4 – Episode 11, 00:07:57
1 Carlos: So can you show me how the money in your in your account is growing, umm.
2 Derek: On that axis?
3 Carlos: By moving your finger along this axis, yeah.
4 Derek: Like starts slow and then just keeps getting faster and faster.
5 Carlos: OK umm and what about the rate of growth?
6 Derek: It would also start slow and keep getting faster and faster.

In Excerpt 4, Derek is engaged in a complex reasoning process that he explains in few words. He imagines that as time passes, the value of his account will increase, and that as the value of the account increases, the relationship between account value and rate causes the rate of change of the account value to increase. Simultaneously, Derek is also imagining that as the rate of change of the account value is increasing, the account is growing "faster and faster" and that as a result, the rate of change, tied proportionally to the account value is also growing "faster and faster."

When I asked Derek to graph this account over the first two seconds, Derek created the graph shown in Figure 13 bottom.

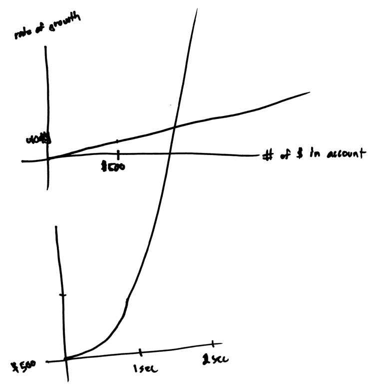

FIGURE 13. Derek's phase plane graph for a continuously compounded account (top) The axes read "rate of growth" and "# of % in account". Derek's graph of the account value over time (bottom).

Derek's graph of the first two seconds (Figure 13) is dramatically different from Tiffany's graph of the first two seconds (Figure 12), in that Derek's graph shows the qualitative behavior of the function, while Tiffany struggled to construct this behavior point by point. Similar to Tiffany, I asked Derek what the graph would look like over a smaller time scale of tenths of a second, and Derek responded that it would look exactly the same, indicating that Derek's "faster and faster" reasoning about exponential growth extended beyond the scope of the graph to any time scale. Later, Derek also extended this way of thinking to create a graph of logistic population growth from the phase plane graph (Figure 14)

8. **Harmonic Waiting Time.** Derek was given this phase plane model in two contexts, first in a financial context, and later as modeling the growth of a population of humans. Early in Derek's work with the financial model, and later when he worked on the population model, Derek imagined a smallest indivisible unit. For the financial model, he imagined that the bank would not keep track of anything smaller than a cent, while for the population model Derek imagined that any result giving a fractional person was unrealistic.

In both cases, Derek resolved this difficulty by imaging that "rate of change" measured the inverse of the waiting time until the next individual was created. So a rate of one dollar per 25 years would mean that the account value would hold constant for a quarter of a year and then increase by one cent. This is in contrast to Derek's differential reasoning in which "rate of change" entailed continuous growth.

Using this way of reasoning, the function that Derek created from the phase plane graph was a step function (Figure 15). Although the graph does not show the

FIGURE 14. Derek's solution to the logistic model with horizontal axis reading "Time yrs". The lower asymptote at was his initial attempt, before placing a scale " "35 Bill[ion]" and "70 Bill[ion]" on the vertical population axis. The upper asymptote showing the classic 'S' shaped curve was his final version.

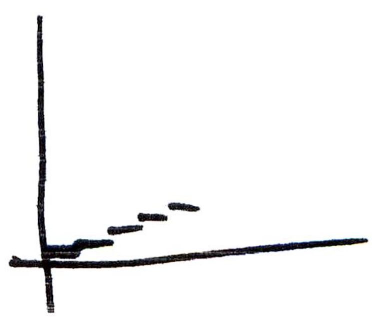

FIGURE 15. Derek's step function solution to the phase plane problem. He described the steps as getting shorter in length, and always changing in by 1 in height

details of Derek's reasoning, Derek was explicit that he imagined that each jump' in the step function was an increase of one in the population, and that the lengths' of the steps go shorter and shorter as the rate increased (and thus the waiting time decreased). As a side note, the resulting step function, based on the harmonic series, does approach exponential growth as time approaches infinity.

Although Derek later accepted the idea of fractional cents, for the population model he found it more valuable to make a distinction between "what was really going on" (the wait time model) and what the equations predicted (the "fantasy world" in which fractional people were permitted), resulting in his rapidly alternating between both models. He sometimes answered the questions I intended for one perspective with the other perspective, as we did not develop a language for specifying which point a view I was asking for.

9. **Discussion.** The students' responses to these tasks, both anticipated and unanticipated, provide a sort of "proof of concept" demonstration that students without access to complex mathematics can still learn and use sophisticated understandings of exponential growth. In their construction of the piecewise linear compound interest function, Both Tiffany and Derek demonstrated a far more sophisticated understanding of the linear assumptions behind compound interest growth than what is typically taught. Derek's qualitative approach to solving exponential and logistic growth in the phase plane mirrors the approaches taught early in differential equations courses. It may seem like a trivial accomplishment from the point of view of someone familiar with this technique, but Tiffany's struggles with the same task show the difficulty of the concepts of continuous change and rate of change that Derek employed. Lastly, Derek's use of waiting time hints at the beginnings of a stochastic processes perspective. Although he would not have had the background in probability necessary to make such a jump, a substitution of a random waiting time for Derek's deterministic waiting time would have had Derek working with a continuous time Markov chain.

The results of this experiment show that these perspectives can be learned in a non-superficial way by the best of students, but this does not mean these perspectives are only attainable by "good" students. Many of the struggles I described these students as having were brought about by the nature of the teaching experiment itself. Initially, the design of the experiment was only intended to teach compound interest and its relation to Euler's method. The methods that Derek in particular employed arose unexpectedly out of his own understanding of continuity and the discussions that we had as a result. In many ways, the understandings that the students came to arose despite of the design of the experiment rather than because of it. The teaching experiment focused entirely on compound interest, and Tiffany's dedication to this idea and resulting difficulties show the insufficiency of this approach on its own. With better design and attention to the approaches Derek used, would be possible to bring these *multiple* sophisticated ways of thinking to a broader variety of students.

10. **Multiple Viewpoints.** Given the trend in standards moving towards a single perspective, a reasonable question to ask would be "are multiple perspectives necessary?" I believe that they are. Over the course of the experiment, Derek used a broad variety of perspectives: geometric, compound, differential, and harmonic. These perspectives were not mathematically consistent. Each approach generated a different graph with different mathematical predictions as to the value of the account at any given time. Despite this, Derek never confused one perspective for another. He switched between perspectives as needed, keeping them separate, but using whatever perspective would better help him understand the situation in the moment. This reflects professional modeling practice, in which a mathematical biologist might refer to the "μ" in $\frac{dy}{dx} = -\mu y$ as the reciprocal of the "average lifespan," despite the fact there is no stochasticity in the equation from which to generate an average. The "average lifespan" comes from the related stochastic model in which lifespans are distributed exponentially.

In contrast, Tiffany's admirable dedication to using only the single perspective of compounding generated results that were far more mathematically consistent, but this consistency came at the expense of understanding. Without the flexibility of Derek's multiple perspectives, Tiffany had some difficulty imagining her calculations as an approximation, and a great deal of difficulty imagining what they were approximating. Without that ability she was unable to take a limit, and she never came to understand continuous exponential growth.

11. **Continuous and Discrete.** Tiffany's inability to construct continuous behavior from discrete reasoning is not particularly remarkable. The emphasis on discrete reasoning and resulting difficulties in graphing tasks are well documented at the elementary and middle level [9]. In their review of literature [9] Leinhardt et al. found that a "pointwise focus" was one of three primary difficulties that students had with graph. More current research shows that these difficulties persist today and to much higher levels of education, including pre-service and in-service secondary mathematics teachers [3, 4, 8, 11, 12, 16] and examples such as Tiffany are showing us that thinking continuously is fundamentally different from thinking discretely but very small. However space is limited. More details on the role of discrete and continuous thinking in Tiffany and Derek's work can be found in previous papers [3, 4].

12. **Concluding Remarks.** Mathematical modelers know that some complex systems are capable of supporting rich dynamics including chaotic behavior – behavior that would be entirely the result of deterministic processes. Chaotic dynamics and prediction are not friendly partners since trajectories are highly sensitive to initial conditions — slight variations in starting population sizes may result in dramatically different life history dynamics. Students are sufficiently complex that what we might initially perceive as "small" variations in understanding might in fact have dramatic consequences later in their mathematical, scientific, or professional lives. As mathematics educators, it is our responsibility to attend to the effects of those "small" initial variations by identifying the different ways (at least as many as possible) that a student develops to understand a subject, and the consequences of those differences in thinking for subsequent understandings. Thompson proposes a similar perspective in the context of mathematics education in general, based on the technique of conceptual analysis [15]. While conceptual analysis asks the question "What mental operations must be carried out to see the presented situation in the particular way one is seeing it?" [7, p. 78], Thompson proposes to use that question "to devise ways of understanding an idea that, if students had them, might be propitious for building more powerful ways to deal mathematically with their environments than they would build otherwise" [15, p. 15]

From a perspective of "working backward" from a goal understandings to the understandings that compose them, I argue that we should discount issues of how exponential growth is traditionally taught, and which version of exponential growth is most compatible with what students have already learned, and instead take seriously the idea that an understanding of dynamical systems should inform (in part) our choices about the earlier education of students. If – purely hypothetically – we want students to understand the stochastic model, then we should push for a more prominent role for probability in the curriculum to prepare students for stochastic exponential growth; while an approach of compound interest from proportionality demands a greater emphasis in early schooling on linear functions; and if we want students to be prepared for differential equations, than a heavy emphasis must be placed on continuity, irrational numbers, and thinking of functions as varying continuously, as demonstrated by Tiffany's difficulties and Derek's success; or we may find, that (as I suspect) we want students to understand more than one of these perspectives.

However, there is the danger that in teaching multiple perspectives, we do not necessarily make clear that these multiple perspectives are separate and contradictory. Students being taught multiple perspectives under the same umbrella term of "exponential growth" may become confused by the contradictory nature of their results, concluding that in certain situations (for example, when certain keywords are present) the student is supposed to use one set of rules, and when other keywords are present, the student is supposed to use a different, contradictory set of rules. An example of this type of problem is the difference between discrete and continuous compounding, where a "rate" of 8% per year[4] means something different, and is placed into a different formula depending on the presence or absence of the word "continuous," despite the fact that both formulas can generate continuous functions. Thus the teaching of multiple perspectives — if adopted — must be done *carefully* and *explicitly*.

With those issues in mind, I put the question to the readers of this article: What do we want – at every educational level – when we say we want students to "understand exponential growth?"

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[4]Note also that the word "rate" in this context refers to a per-cent or per-capita rate, which is a very different type of rate than the slope-of-a-line rates that students would have seen previously, and its connection to the slope of the line is not-obvious. Thus "rate" actually means *three* different and contradictory things in this situation.

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