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Chunky and smooth images of change

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Imagine a bottle being filled with liquid [1]. How do you think about the volume of the liquid as it changes relative to the height of the liquid in the bottle. One way is to envision sections along the height of the bottle and determine/estimate amounts of volume in each section. Alternatively, one might envision both the volume and the height changing together so that each is *continually* increasing. The former way of thinking could be likened to filling a bottle with successive cups of liquid: changes in volume and height occur in discrete chunks. The latter way of thinking could be likened to filling the bottle from a hose: changes in volume and height continually progress. These different ways of thinking indicate how students might draw on differing images of change when constructing relationships between changing quantities (in this case, the volume and height of water in a bottle).

Imagery has played a prominent role in the study of change. The work of fourteenth century scholar, Nicole Oresme, drew on images of change as a continuous, measurable quantity: “Everything measurable, Oresme wrote, is imaginable in the manner of continuous quantity” (Boyer, 1991, p. 264). Oresme thought of change as an intensity of a measurable object, indicating a line to be a “fitting” representation of an intensity, because both intensities and lines could be increased or decreased without bound (Clagett, 1968). Oresme gave examples of what he meant by intensity: “intensity is that according to which something is said to be ‘more such and such,’ as ‘more white’ or ‘more swift’” (Clagett, 1968, p. 167). Although Oresme did not precisely define intensity, his use of the term suggests he envisioned the nature of intensity to be that of a continuum (Clagett, 1968). Oresme’s continuous images of change were instrumental in advancing the study of change, hinting at later uses of graphical representations of varying quantities (Edwards, 1979).

What images might be entailed in envisioning change as being capable of varying in intensity? Might student-generated representations provide windows into their images of change? Might images of change as continually progressing (e.g., filling a bottle with a hose) also entail images of change as occurring in discrete chunks (e.g., filling a bottle with successive cups of liquid)? We argue that considering students’ images of change provides insight into how students conceive of variation and make sense of situations involving varying quantities. In this article, we posit two contrasting images of change—*chunky* and *smooth*, provide empirical examples to support the different images, and articulate the affordances and constraints of each. These different images of change have implications for students’ conceptions of variation and of situations involving varying quantities.

Conceptualization of variation

An individual’s conceptualization of variation involves creating images of change. By *conceptualization*, we mean engaging in mental operation (Piaget, 1970). An operation is a mental activity that an individual is capable of executing without necessarily engaging in observable action. For example, an individual can envision how the volume and height of water might vary when a bottle is being filled without physically filling the bottle. We use image to mean the “dynamics of mental operations”

(Thompson, 1994a, p. 231). Therefore, different images of change involve and promote different mental operations.

By considering students' images of change in situations involving variation, several researchers have investigated students' understanding of concepts such as rate of change/differentiation, limit, and accumulation/integration (*e.g.*, Carlson *et al*., 2002; Kidron, 2011; Oehrtman *et al*., 2008; Rasmussen, 2001; Thompson, 1994a; Thompson, 1994b; Zandieh, 2000). Collectively, this research suggests that the mechanics of how students conceptualize variation influences the mathematics that they construct. Drawing from our work with secondary school (Castillo-Garsow, 2010; Johnson, 2012a, 2012b) and university students (Moore, 2012), we make distinctions between students' images of change that appear to involve different mental operations.

A task situation and two students' images of change

The use of the terms *chunky* and *smooth* to describe students' images of change emerged from a teaching experiment conducted by Castillo-Garsow (2010). Castillo-Garsow drew parallels between linguistic time research (Bohnemeyer, 2010) and covariation research (*e.g.*, Confrey & Smith, 1994; Saldanha & Thompson, 1998) to (i) characterize differences between path and sequence metaphors for time in linguistics (Bohnemeyer, 2010) and (ii) make distinctions between Confrey and Smith's (1994) and Saldanha and Thompson's (1998) approaches to characterizing covariation. Results of Castillo-Garsow's (2010, 2012) research include models of high school algebra students' mathematical thinking. The students on whom the models were based, "Derek" and "Tiffany," worked on tasks involving a variety of bank account policies. For each policy, the students were to develop graphs and equations that described the behavior of an investment under the policy.

To illustrate Tiffany and Derek's work, we draw from the portion of the teaching experiment when students were considering the behavior of an investment over time. The investment policy was posed as follows:

> The Savings Company (SayCo) also competes with Yoi Trust and Jodan. SayCo's PD8 account policy is as follows: if you have one dollar in your bank account, you earn interest at a rate of 8 cents per year. For each additional dollar, your interest increases by another 8 cents per year. If you have fractions of a dollar in your account, your interest increases by the same fraction, so 50 cents earns interest at 4 cents per year. Here is SayCo's new feature: at any moment you earn interest, SayCo adds it to your account balance; every time your account balance changes, SayCo pays interest on the new balance and calculates a new growth rate. Why is SayCo's PD8 the most popular account?

The students were asked to create a graph depicting rate of growth of the account in dollars per year in relation to the value of the account in dollars, and then use that graph to create a new graph depicting the value of the account in dollars in relation to time. In this section, we discuss Tiffany and Derek's work primarily on this latter graphing task.

Tiffany

Tiffany sketched a graph relating account value and time such that the dollar per-year rate was proportional to amount (Figure 1).

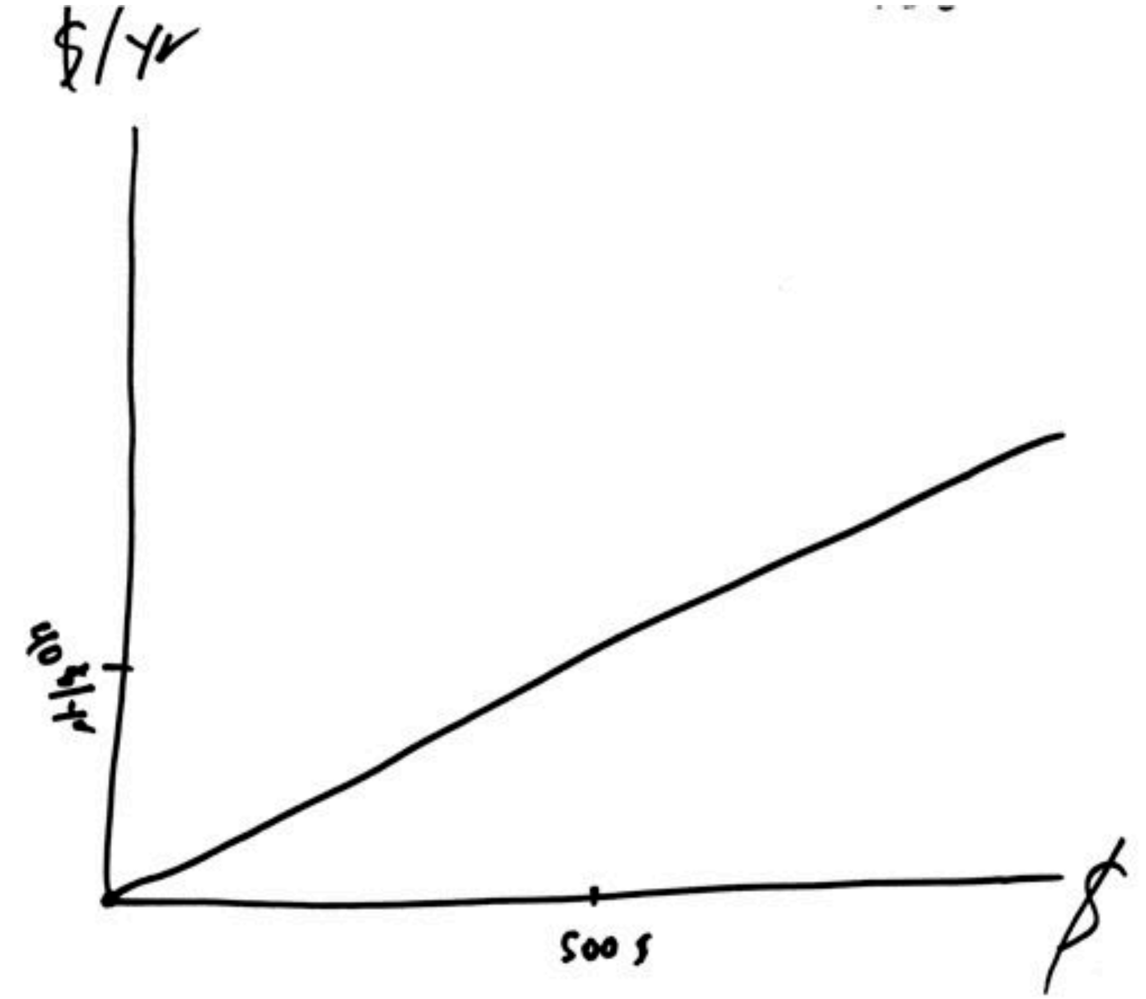

Figure 1. *Tiffany's graph showing the proportional relationship between the dollar per year growth rate of an account and the value of the account.*

While creating her graph relating account value and time, Tiffany imagined time passing in equal-sized chunks based on a standard time unit (*e.g.*, seconds, tenths of seconds, *etc*.). She reasoned that as time changed in equal-sized chunks, each successive change in the account value would be larger than the one before. Initially, Castillo-Garsow asked Tiffany about chunk sizes of one second, but then attempted to direct Tiffany to chunk sizes of smaller than one second:

Carlos: Now tell me what happened during that second.

Tiffany: That second during that second you earned a little bit of money so now they're gonna recalculate and stuff for your new-

Carlos: OK,

Tiffany: -rate.

Tiffany imagined "what happened during that second" as someone doing the calculations needed to find the value of the account at the completion of that second and then determining the rate for the next second. However, Castillo-Garsow intended this question to be about "what happened" *to the account within* that second, a perspective that he had been attempting to help Tiffany develop by asking her about the values of accounts at successively smaller intervals of time. Castillo-Garsow subsequently drew Tiffany's attention to the account value at one tenth and one thousandth of 1-second intervals, and he then asked her to create a graph of the function for the first 3 seconds. Tiffany chose a chunk size of 1 second, and scaled the graph so that the points were clearly distinguished.

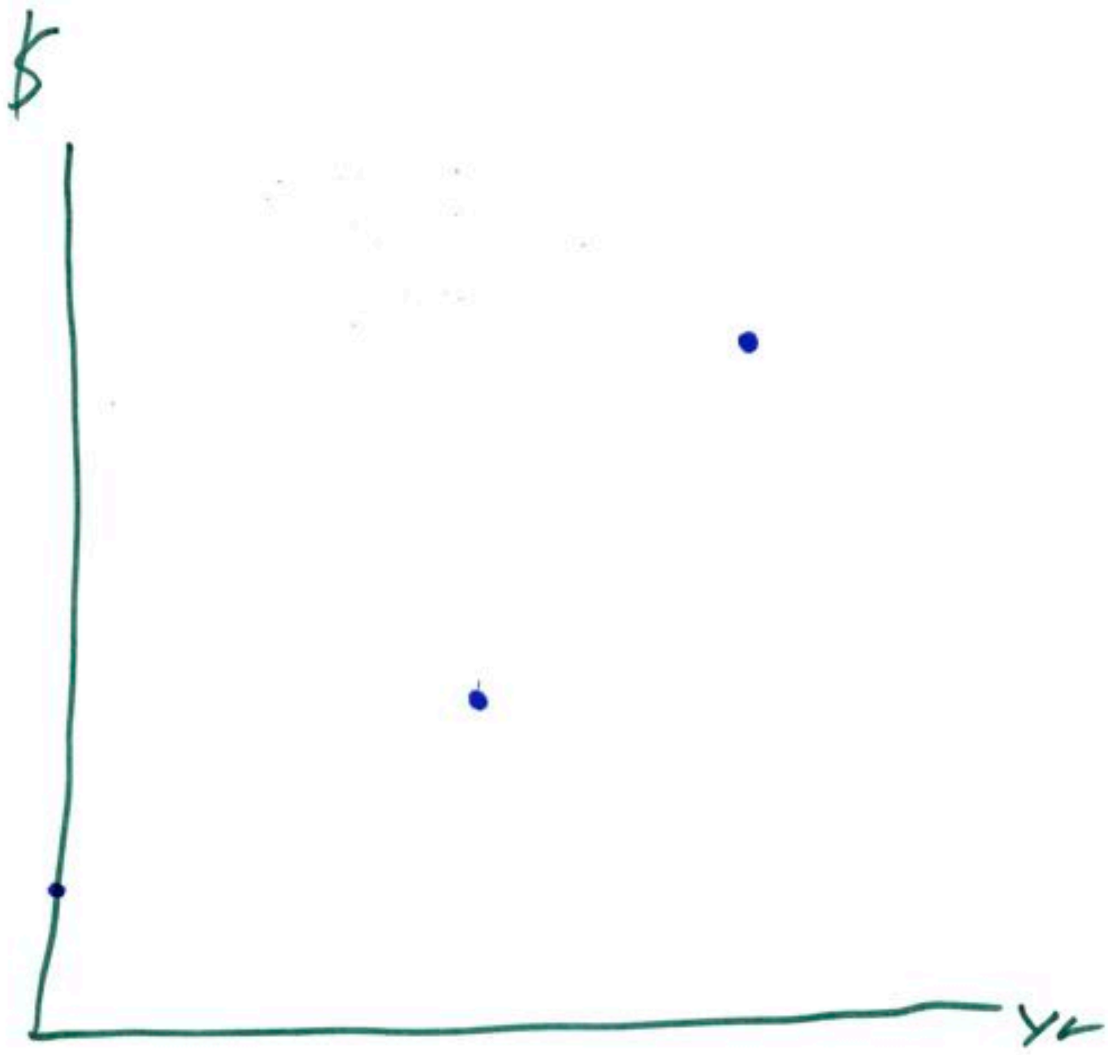

Figure 2. *Tiffany's solution to the phase plane problem. The 3 points marked are each one second apart while the horizontal axis reads "yr" for years (from Castillo-Garsow, 2010, p. 155).*

The contrast between Tiffany's continuous line graph in Figure 1 and the point graph she drew in Figure 2 can be explained by the different thinking used to create each graph. Tiffany created the line graph in Figure 1 from an equation that she recognized as the equation of a line, and drew the shape that she associated with that equation (Castillo-

Garsow, 2010). While creating the graph shown in Figure 2, however, Tiffany did not have access to an equation or know that the graph would be exponential. Without that information, Tiffany based her graph (Figure 2) on her understanding of the situation: that each successive value must be calculated (a process that takes some non-zero amount of time), and that after each (imagined, not necessarily carried out) calculation, the result would be an account value that changed by more than the previous change in account value. This reasoning led her to construct a graph consisting of separate points, a separation that was important to Tiffany. Although the horizontal axis was scaled in years, Tiffany did not place the three points close together, near the origin, but rather emphasized the distinctiveness of each point by spreading them across the page. Choosing a 1 second chunk size is perhaps natural from the point of view of a question about "3 seconds." However, there are multiple ways of interpreting the task of graphing over 3 seconds.

Derek

Another interpretation of the task of "graphing over 3 seconds" can be found in Derek's work. Whereas Tiffany imagined change in terms of one-second chunk sizes, Derek imagined change as occurring *smoothly*:

Derek: What I don't get is like it does it if you have the dollar and then you just put in a dollar at any point does it change right away?

Carlos: Umm well it seems like the way this was written that if you put in another dollar it would change right away. But let's imagine for a moment, umm, that we’re only putting money in this account one time. We’re only investing in this account one time.

Derek: So does the eight cents interest also affect it? The change of growth rate?

Carlos: Umm I’m not sure what you mean by that.

Derek: Like if like if you have a dollar you put in a dollar and the growth rate changes right away? Well if you’re getting money constantly is the growth rate increasing constantly?

Following his explanation, Derek sketched a graph in two chunks, completing a section for each second. Because his chunks resulted from smooth images of change (*e.g.*, “growth rate increasing constantly”), he produced a graph that is a continuous curve (Figure 3, bottom).

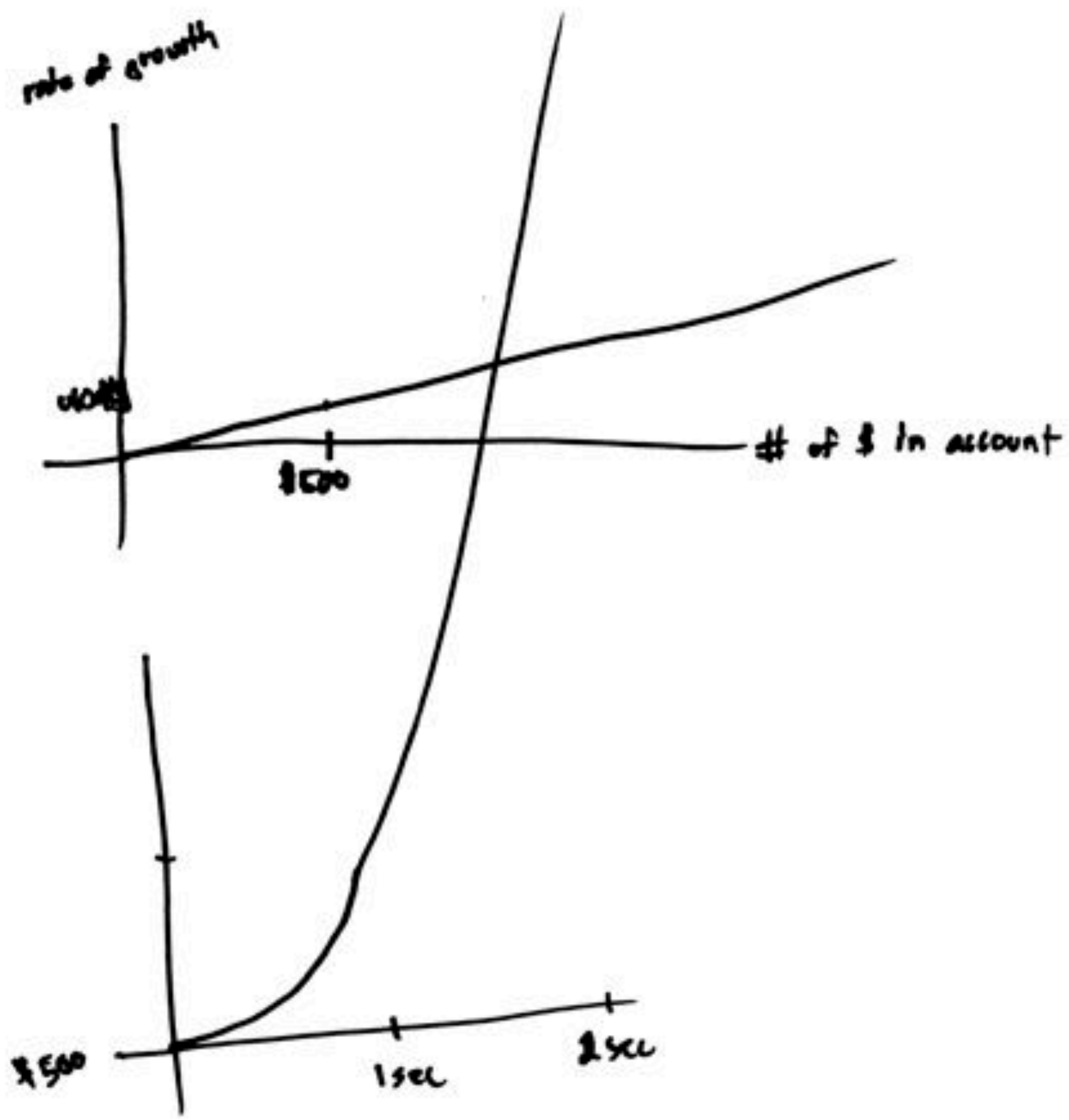

Figure 3. *Derek's graph of the account in the phase plane (top) and his solution to the phase plane problem (bottom) (from Castillo-Garsow, 2010, p. 172).*

Looking across Tiffany and Derek's work

Although Tiffany and Derek drew relatively similar graphs relating rate and account value, their graphs relating account value and time were drastically different. This outcome can be explained by their differing images of change. When creating the graph, Tiffany conceived the account value as changing *after* a specified interval of time. In contrast, Derek conceived the account value as changing *with* continuous changes of time. Just as a student can imagine volume and height accumulating in a bottle in different ways (*e.g.*, filling cup by cup or with a hose), the work of Tiffany and Derek, respectively, suggest two distinctly different images of change: *chunky* and *smooth*.

Chunky images of change

Chunky images of change involve imagining change as occurring in completed chunks (Castillo-Garsow, 2010, 2012). Two features characterize chunky images of change: a unit chunk whose repetition makes up the variation, and the lack of an image of variation within the unit chunk. Therefore, ongoing change is generated by a sequence of equal-sized chunks, and this makes measuring change essentially about counting how many chunks have occurred. A key aspect of the chunk is that it is conceived of atomically. Chunky thinking is thinking *in* intervals, but it is not thinking *about* intervals. Therefore, the students' attention is on what seemingly occurs at the edges in an interval, *after* an imagined interval of change has elapsed.. The intermediate values within the chunk "exist," in the sense that they are needed to fill out the chunk, but they receive little or no attention. For example, even when Castillo-Garsow asked Tiffany to talk about what happened *during* 1 second, she interpreted the question in terms of calculating the account value *at particular instances of seconds*.

Because an individual using chunky thinking works in atomic units, she reasons in discrete points using the resolution of that unit. We refer to the spaces between these discrete points as *holes.* By holes, we do not mean that nothing exists in the spaces, but instead emphasize that a key factor in chunky thinking is an essentially singular focus on discrete points that exist at the edges of the chunks. Both the size of the chunk and the scale of a representation can illustrate the inherent nature of holes in chunky thinking. In Tiffany's case, instead of continuing with a chunk size of one thousandth of 1 second or creating a graph on a scale of years (both of which would obscure the holes), she chose a chunk size and scale that made the holes explicit. No matter how small she cut up her chunks, Tiffany always conceptualized them as chunks, and imagined them at a scale where she could see the chunkiness.

The place of holes in Tiffany's thinking also points to the asynchronous nature of change for an individual using chunky thinking. In Tiffany's experience, the event "finding the value of the account at 1 second" happened *before* the event "finding the value of the account at one tenth of 1 second," because the value at 1 second was needed to calculate the value at one tenth of 1 second. More generally, it was necessary that Tiffany imagined what happened at the edges of a chunk *before* she could consider what happened *during* the chunk. When time is imagined in terms of chunks, a mismatch exists between a temporal ordering of values within a situation and an experience of those values by a student doing the imagining. This mismatch makes it seemingly impossible for a student using chunky thinking to imagine a situation dynamically while simultaneously imagining the mathematics of it.

When an independent variable varies in chunks, the dependent variable can be calculated from the independent variable. Such an approach is commonly used by U.S. students when graphing functions: students plot a few points in order to get a sense of the function and then connect the points. Leinhardt, Zaslavsky and Stein's (1990) literature review on graphing criticized in detail instruction that supports this method, and we argue that this type of graphing is not necessarily *covariational* (Carlson *et al*., 2002; Confrey & Smith, 1994; Saldanha & Thompson, 1998). Rather, it can be variational, as plotting points only requires imagining x varying, and only in discrete ways. Because the space between each point remains unevaluated, this pointwise approach to graphing supports a chunky conceptualization of variation and vice-versa.

Although a pointwise approach enables students to graph a multitude of functions, an issue arising with this approach is that a chunk size needs to be determined prior to graphing, often by a secondary understanding of the problem. An example of this issue emerges in the work of "Bob," an undergraduate student enrolled in a secondary mathematics content for teaching course (Moore, 2012). Bob was prompted to draw a graph of $y = \sin(3x)$. He first gained a sense of the function by plotting points for x-values of 0, $\pi/2$, and π. After plotting these points, Bob concluded, "I guess it just reflects it [sin(x)]," and drew the graph shown in Figure 4. Bob's use of a chunk size of $\pi/2$ would have been a appropriate were he graphing $y = \sin(x)$. However, because the chunk size that he chose aligned with critical points of $y = \sin(3x)$, the graph appeared to him to be a trigonometric curve with period 2π rather than period $2\pi/3$.

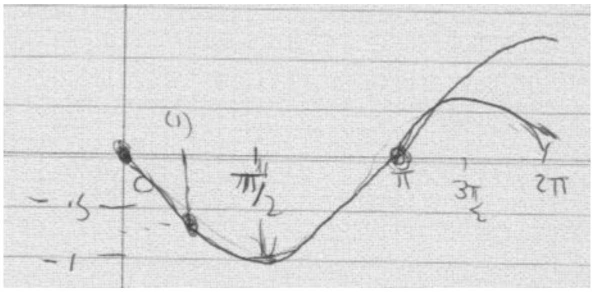

Figure 4. Bob's graph of y = sin(3x) (from Moore, 2012). The point at x=1 was added later.

Although reducing the size of the chunk is sufficient to address Bob's specific situation, no chunk size is sufficient to cover all situations dealing with trigonometric functions. If Bob had adopted a standard of selecting a chunk size of π/6 instead of π/2, he would have correctly graphed the function $y = \sin(3x)$. However, to graph $y = \sin(9x)$, an even smaller chunk size is needed. And although a student using chunky thinking might be able to reduce the size of a chunk, the student would still be imagining the change in chunks (*e.g.*, Tiffany's use of 1 second chunks, then one tenth of 1 second chunks). Therefore, reducing the chunk size is not sufficient because no matter how small the chunk size, the focus remains on discrete points. In order to move beyond a "pointwise focus" described by Leinhardt, Zaslavsky & Stein (1990, p. 45) or the chunky thinking that promotes it, some other way of thinking seems necessary to make sense of change that might be occurring within a chunk and to guide the selection of chunk size.

Smooth images of change

Smooth images of change involve imagining a change in progress (Castillo-Garsow, 2010, 2012). Ongoing change is generated by conceptualizing a variable as always taking on values in the continuous, experiential flow of time. A smooth variable is always in flux. The change has a beginning point, but no end point. As soon as an endpoint is reached, the change is no longer in progress.

Smooth images of change are not the same as chunky images of change cut up really small. Smooth images of change involve an entirely different conceptualization of variation. The difference may best be illustrated with a pair of metaphors. First, imagine a child throwing a ball through a paper wall. Before throwing the ball, the child anticipates that the wall will be broken. The child does not have to slice time into smaller and smaller intervals or take a limit to determine that she cannot throw a ball through a paper wall without damaging it, no matter how thin the paper is. The child knows this because she experiences continuous motion from infancy; the ball must come into contact with the wall because the wall is in the way of the ball's motion.

Contrast this with a second metaphor, an animated movie of a ball thrown through a paper wall. Smooth action appears to occur when watching the movie. Behind the scenes, however, a movie is a series of still images played at 24 frames per second. There is no sense of motion when looking over each image by hand on a light box. The ball depicted on film appears at a fixed location in each frame and it is possible that there is no frame where the imaginary ball overlaps the wall. Also, the animator can choose whether the paper wall is torn or not.

Now, imagine the child watching the animated movie. Drawing from her own physical experience, the child might imagine the ball colliding with the wall. However in the reality of the animator, the ball never collided with the wall, because none of the frames he drew show the ball in contact with the wall. This difference in the child's reality (relating the ball to physical experience and perceiving the movie in continuous motion in progress) and the animator's reality (individual frames in which there is no motion) is analogous to the difference between smooth and chunky images of change.

We also note that the "true" nature of the reality itself does not matter. The child uses smooth thinking in both the real ball and animated ball cases by imagining the ball in continuous motion. Similarly, the animator, who imagines the animated ball as individual frames, could also imagine the real ball in frames by imagining change at the molecular, atomic, or quantum level. Images of change are conceptualizations and are thus in the mind of the beholder.

In our earlier example, Derek solved the graphing task by imagining change in progress, analogous to watching a movie *without* ever thinking about the frames or taking a limit. In contrast with Tiffany, it was *not* thinking about the frames that was *key* to Derek's solution. This is not to say that Derek could not imagine change as occurring in small chunks. In fact, when Derek did think about small chunks (such as individual people in populations of billions), he zoomed in (like Tiffany) to make discontinuities visible (Castillo-Garsow, 2010). But unlike Tiffany and Bob, Derek imagined what happened *during* the chunk *before* determining the ending edge of the chunk. With Bob, we argue that if the edges of a chunk come first then an individual's use of smooth thinking can be constrained by the predetermined amounts.

Using smooth thinking to consider chunks

Although smooth thinking and chunky thinking are fundamentally different, they are not unrelated. We find it difficult to conceptualize how chunky thinking might be a cognitive root (Tall, 1989) for smooth thinking (*e.g.*, no matter how small the chunks,

chunks are always present). However, we argue that if smooth thinking were a cognitive root for chunky thinking, it might form a foundation for a powerful form of chunky reasoning.

Consider Hannah, a tenth grade student, who responded to an adaptation of the bottle problem described at the start of this article. The adaptation, developed by Johnson (2010), required Hannah to sketch a viable bottle shape that could be represented by a nonlinear graph representing the volume of liquid as a function of the height of liquid in a filling bottle. When drawing a viable bottle shape, Hannah considered different sections of the bottle, imagining the height as increasing continuously. Unlike students using chunky thinking, who based their sections on particular amounts of volume and height (Johnson, 2012a), Hannah based her sections on when the volume of filling liquid would increase at different intensities (*e.g.*, a faster, slower, or "steady" increase) with respect to the increasing height (Johnson, 2012b). Hannah's description, along with her gestures, indicated her imagining a change in progress from an initial state of an empty bottle, suggesting that she was using smooth thinking to make sense of how the height and volume of liquid were covarying. Although Hannah's sectioning of the bottle created chunks, the chunks came as a consequence of smooth thinking. Therefore, Hannah's chunks did not entail holes because variation within the chunks was at the forefront of her conception of the situation.

Another example of a student using smooth thinking emerges in the work of "Zac," an undergraduate precalculus student interviewed by Moore (2010, 2012). Zac was prompted to draw a graph of the distance of a passenger from the ground in relation to the time elapsed on a Ferris wheel ride (for a wheel radius of 36 feet):

Zac: Ok. So a really easy way to do this is divide it up into four quadrants (*divides the circle into four quadrants using a vertical and horizontal diameter, Figure 5*). ‘Cause we’re here (*pointing to starting position*), for every unit the total distance goes (*tracing successive equal arc lengths*), the vertical distance is increasing at an increasing rate (*writing i.i.*)...Then, uh, once she hits thirty-six feet, halfway up, it’s still increasing but at a decreasing rate (*tracing successive equal arc lengths, writing i.d.*)...Uh, then when she hits the top, at seventy-two, it’s decreasing at an increasing rate (*tracing successive equal arc lengths, writing d.i.*)...And then when she hits thirty-six feet again it’s still decreasing (*making one long trace along the arc length*), but at a decreasing rate (*tracing successive equal arc lengths*, *writing d.d.*)

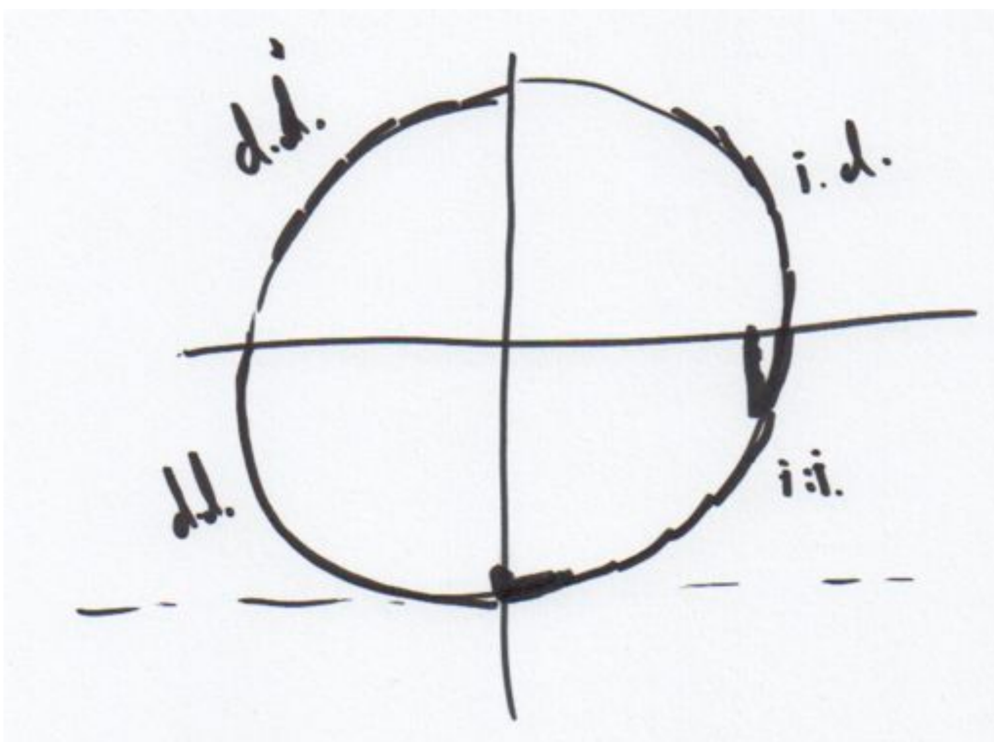

Figure 5. Zac’s diagram of the Ferris wheel (from Moore, 2012).

Zac's actions of cutting up the Ferris wheel into quadrants and tracing successive arc lengths suggest chunky thinking. However, his tracing arc lengths and terminology (increasing, increasing rate, and the use of the present tense for variation) indicate imagining change in progress, a hallmark of smooth thinking. To Zac, his chunks and subchunks contained smooth variation, and he focused on such variation to get a sense of how the quantities covary. Furthermore, like Hannah, the imagining of change in progress from the beginning of the ride seemed to support Zac's consideration of variation in the intensity of the rate.

It is important to note that we do not consider a person to use chunky thinking or smooth thinking (or combination of the two) in absolute terms. Rather, we use smooth thinking and chunky thinking as "in the moment" classifications to describe an individual's thinking related to a particular image of change. However, we argue that differences in the interplay between these images of change and their place during a student's solution have mathematical implications.

Hannah and Zac illustrate students productively engaging in a combination of smooth and chunky thinking. In contrast, Bob illustrates a student whose orienting acts (initial making sense of a problem (Carlson & Bloom, 2005)) were limited to chunky thinking. By using a chunk size of $\pi/2$, he determined particular values that formed parameters for his envisioning a prototypical shape all at once in order to "fill in" the chunks. As such, his drawing of

the graph was not a product of smooth thinking, but rather a replication of previously experienced shapes. Later in the interview, Bob used the produced shape as data for imagining values within his chunks. Bob's process is similar to the asynchronous nature of change in Tiffany's case, which stands in contrast to *first* reasoning about emergent changes in progress to *anticipate* a shape (*e.g.*, Zac). By first imagining a smooth increase from $x = 0$, one can imagine $\sin(3x)$ increasing at a decreasing rate until an input value of $\pi/6$ (*e.g.*, an argument value of $\pi/2$). Orienting to a chunk size and producing products based on this chunk size can form constraining parameters when imagining a change in progress post-orientation. First imagining a change in progress, however, can lead students to determine an appropriate chunk size.

Mathematical differences in chunky and smooth thinking

A key distinction between smooth and chunky thinking lies in the nature of the product generated as a result of the thinking. Simply put, chunky thinking generates chunky conceptions of variation, whereas smooth thinking generates smooth conceptions of variation, with these conceptions producing different mathematics. Although a smooth conception of variation involves imagining a changing number or magnitude, it also involves attending to all states continuously without privileging any sort of unit value that would form the basis of counting. Therefore a smooth conception of variation does not immediately generate products that could be counted. In contrast, products of a chunky conception of variation are always countable, because no matter how small the chunk, discrete points always exist and can be counted.

It was smooth images of change that supported Oresme's consideration of variation in the intensity of change. Appealing to an imagined point moving along a line, Oresme indicated three distinctly different types of change: a particular intensity throughout, a “regularly” increasing or decreasing intensity, and an “irregular” motion that could not be accounted for by regular motion (Clagett, 1968). The different intensities distinguished between what would later be referred to as constant and varying rates of change, suggesting at least two types of varying rates of change. Oresme’s work influenced the development of Leibniz’s symbolic representations of instantaneous rate of change that emerged in the seventeenth century, giving rise to a calculus based on smooth images of change (Edwards, 1979; Kaput, 1994). It was only with the advent of modern analysis that a fully chunky (epsilon-delta) calculus was possible—a calculus in which limits are characterized in measurable intervals (chunks) without necessitating an appeal to the non-chunky idea of “infinitesimal” or ideas of smooth motion such as “approaching” in proof. Still, during the development of this calculus, mathematicians relied on smooth thinking to develop intuitions and conjectures before writing the formal, chunky proofs. Since we ask students to reason about change and rate long before they study analysis, it seems reasonable that beginning with smooth images of change could create the foundations for students’ consideration of the difficult-to-learn concepts of limit, rate of change/differentiation, and accumulation/integration.

Concluding remarks

We argue that smooth images of change are more powerful than chunky images of change, in part because of the mathematical difficulties that students using chunky thinking (*e.g.*, Tiffany and Bob) seem to have that students using smooth thinking (*e.g.*, Derek, Hannah, and Zac) do not have. Our argument is also philosophical, however: because it seems impossible for an individual using smooth thinking to experience imagining an ongoing change forever, we conjecture that at some point, the imagining of ongoing change must stop, at which point an individual using smooth thinking would have imagined a completed change (a chunk). Therefore, smooth thinking would entail a capacity to think in chunks (or, at least at its foundation, one chunk). In contrast, in our experience with students, chunky thinking does not seem to entail a capacity to think smoothly, nor does chunky thinking seem to provide a cognitive root for smooth thinking. These musings remain open for future inquiry. Particularly, we suggest that future research explore how students develop smooth and chunky thinking and possible relationships between the two (*e.g.*, how might students using chunky thinking can be encouraged to using smooth thinking?). Investigating the consequences of smooth and chunky thinking across various mathematical topics might also provide elaboration on characterizations of smooth and chunky images of change and their role in mathematical thinking and learning.

Notes

[1] The bottle problem was developed by the University of Nottingham's Shell Centre (Swan & the Shell Centre Team, 1999).

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