Teaching the Verhulst Model:

A Teaching Experiment in Covariational Reasoning

and Exponential Growth

by

Carlos Castillo-Garsow

A Dissertation Presented in Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

ARIZONA STATE UNIVERSITY

August 2010

Teaching the Verhulst Model:

A Teaching Experiment in Covariational Reasoning

and Exponential Growth

by

Carlos Castillo-Garsow

has been approved

July 2010

Graduate Supervisory Committee:

Patrick Thompson, Chair
Marilyn Carlson
Michael Oehrtman
Luis Saldanha
Sergei Suslov

ACCEPTED BY THE GRADUATE COLLEGE

ABSTRACT

Both Thompson and the duo of Confrey and Smith describe how students might be taught to build “ways of thinking” about exponential behavior by coordinating the covariation of two changing quantities. However, these authors build exponential behavior from different meanings of covariation. Confrey and Smith advocate beginning with discrete additive and multiplicative changes, while Thompson advocates beginning with continuous variation. In light of these differences, this work investigates the questions of how students actually reason covariationally, and what the consequences of their reasoning are for mathematics involving exponential functions.

This work describes a teaching experiment, consisting of a series of fifteen task-based exploratory teaching interviews with two high school student participants. The purpose of the experiment was to identify the operations of covariational reasoning that the students actually used, and the consequences of that reasoning for mathematics involving exponential growth. The tasks covered linear functions, compound interest, phase plane, exponential growth, and the logistic differential equation.

In the conceptual analysis of the participants’ mathematical work and spoken utterances, I identified two ways of thinking about change that differ from the discrete/continuous dichotomy above: thinking about “chunky” completed changes, or a “smooth” change in progress. With smooth and chunky as a basis, I also identify five different ways of understanding exponential growth: Geometric, compound, differential, harmonic, and stochastic. Lastly, I suggest that powerful understandings of exponential growth come not from the mastery of any one way of thinking, but from a rapid and fluent shifting amongst several ways of thinking.

ACKNOWLEDGMENTS

Without the support of a great number of people, this dissertation could never have come into existence. First and foremost, I would like to thank Patrick Thompson for his incredible mentorship and uncompromising standards of scholarship. His comments and questions have always been both terrifying and of immeasurable value. Special thanks also to Debra Castillo, for volunteering her editing expertise to make sure that this work approaches legibility. All remaining errors are mine, not theirs.

At a more personal level, I would like to thank Carlos Castillo-Chavez, Wolfgang Wölck, Justin Boffemmeyer, and Uri Treisman, who at various points in my life, all helped me get back up after I had stumbled. Without the support of these excellent individuals, I never would have started a dissertation study at all. Lastly, I would like to thank Marilyn Carlson for keeping me in food, rent, and good humor while I wrote this.

Research reported in this dissertation was supported by National Science Foundation Grant No. EHR-0353470, with Patrick Thompson as principal investigator. Any conclusions or recommendations stated here are those of the author and do not necessarily reflect official positions of NSF.

TABLE OF CONTENTS

CHAPTER Page

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure Page

Figure Page

Figure Page

Figure Page

Figure Page

Figure Page

CHAPTER 1

STATEMENT OF THE PROBLEM

Mathematical modeling has been gaining increasing prominence in mathematics education. The National Council of Teachers of Mathematics standards for algebra at the primary and secondary level includes "use mathematical models to represent and understand quantitative relationships" (NCTM, 2000, p. 303); while the Mathematical Association of America (MAA) Committee on the Undergraduate Program in Mathematics (CUPM) places a heavy emphasis on "[Communicating] the breadth and interconnections of the mathematical sciences" (Barker, et al., 2004, p. 6) and "[Promoting] interdisciplinary cooperation" (p. 7). In the course of a 103 page document, the CUPM committee mentions "modeling" 44 times. All but one of these uses of "modeling" refers to mathematical modeling. The Curriculum Renewal Across the First Two Years (CRAFTY) subcommittee of CUPM recommends that instructors of early undergraduate mathematics courses "emphasize mathematical modeling" as one of their summary recommendations (Barker & Ganter, 2004a).

However mathematical modeling is not a single unified subject. CUPM makes a brief list of forms of mathematical modeling that instructors should be familiar with, including "differential and difference equations, linear statistical models, probability models, linear programming, game theory, and graph theory" (Barker, et al., 2004, p. 56). Each of these forms requires different techniques and different ways of thinking. The ideas behind local stability analysis of systems of differential equations are very different from the ideas behind regression in data fitting or the ideas of propositional logic used when modeling philosophical argument. Even within a single field, such as mathematical biology, mathematical modelers vary wildly in their philosophies, goals, and techniques (Smith, Haarer, & Confrey, 1997).

In order to make sense of how students engage in mathematical modeling, each modeling field, modeling technique, and modeling philosophy must be explored in detail. Previous studies in mathematical modeling (Burghes, 1980; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Hestenes, 2006; Lesh & Doerr, 2003; Shoenfeld, 1991) have focused largely on modeling in a generic sense, on the student's process of increasing formalization and abstraction of a situation or problem. This approach certainly has merits, but it does not address the details of particular models: What specific ways of thinking do students need in order to become fluent in specific models, and specific modeling techniques?

In the area of dynamical systems modeling, Several researchers have studied ways in which students might operationalize reasoning about relationships between changing quantities, called covariational reasoning – see (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1994, 1995; Saldanha & Thompson, 1998; Smith & Confrey, 1994; Thompson, 1994a, 1994b, 2008b). However, these researchers have come to very different meanings of covariation and these different meanings result in different connections among covariational reasoning, constant rate of change, and exponential reasoning (Confrey & Smith, 1994, 1995; Thompson, 2008a). It is important to flesh out these differences because students who use the mental operations described by these authors will approach and understand function and rate and exponential growth in very different ways.

Understanding exponential functions is critical to mathematical modeling in a variety of fields. The CRAFTY curriculum foundation project based its curriculum recommendations on interdisciplinary input in a number of fields. Of the eighteen fields that CRAFTY reported on, a total of fourteen gave exponential functions as a relevant mathematical topic: Biology, Chemistry, Computer Science, Chemical Engineering, Civil Engineering, Electrical Engineering, Mechanical Engineering, Health-Related Life Sciences, Interdisciplinary Core Mathematics, Physics, Statistics, Teacher Preparation, Biology and Environmental Technology, and Electronics, Telecommunications and Semiconductor Technology (Barker & Ganter, 2004b). The remaining four were Business and Management, Mathematics, Information Technology, and Mechanical and Manufacturing Technology, all fields in which exponential functions are still valuable, even if exponential functions were not mentioned explicitly in those reports.

Although there appears to be near universal agreement that exponential function understanding is critical in mathematics-related fields, none of the CRAFTY recommendations describe what ways of understanding exponential functions are most useful to their students. In fact, the report seems to take it for granted that there are commonly accepted goals and standards for exponential function understanding that are universal across all fields and all researchers. As an example, Huff and Terrell write in their report: “Students should have a mastery of exponential functions (including base e exponentials), and logarithm functions, including properties of these functions required for their successful implementation in problem-solving" (Huff & Terrell, 2004). But Huff and Terrell do not describe the properties of the exponential to which they are referring. They seem to take for granted that there is a common agreement on what aspects of exponential functions are important and useful for students to learn. However the work of Confrey and Smith (Confrey & Smith, 1994, 1995; Smith & Confrey, 1994) and Thompson (Thompson, 2008a) show that this is not the case. These two authors present different ways of thinking about exponential growth, and those different ways of thinking about exponential growth are based on different ways of thinking about covariation and rate of change.

Over the past two and a half years, the Teachers Promoting Change Collaboratively (TPCC) research project has engaged in a classroom intervention, following students as they progress as a class through Algebra I, Geometry, and Algebra II classes. The researchers have worked in collaboration with the teachers of these classes on a curriculum that emphasizes covariational reasoning and mathematical modeling. Originally, the purpose of this intervention was to provide cases of teachers engaged in instruction to be used for teacher professional development; however, this environment also provides an ideal opportunity to study the meanings of covariation the students have developed as a result of their interactions with the course.

Specifically, this teaching experiment examines the consequences of those meanings for exponential reasoning and mathematical modeling by engaging the students in challenging problems: the development of the exponential model from first-principles (Smith, et al., 1997)), and an investigation of the Verhulst model – one of the earliest adaptations of the exponential model in population modeling (Verhulst, 1977). The Verhulst model was selected because it is commonly used as first model for introducing ideas of population modeling (Bergon, Harper, & Townsend, 1996; Brauer & Castillo-Chavez, 2000; Edelstein-Keshet, 1988).

Simply put, the questions that I had hoped to answer with this teaching experiment were: When engaged in instruction that emphasizes covariational reasoning, what meanings of covariation do students actually develop, and what are the consequences of those meanings for students' understanding of what is conventionally taken as dynamical systems modeling? Those questions are answered in part. I will show that the different ways in which students operationally think about change can have a dramatic impact in the students' understandings of how an exponential function behaves, and the reasons that students give for those behaviors. However, as a result of this teaching experiment, my own meaning of "a good understanding of exponential growth" has changed.

This is a story of two students, whose different ways of thinking about change resulted in very different understandings of exponential growth, but it is also a story of the development of my own understanding of exponential growth as a result of working with these two students. Over the course of this study there have been subtle changes in the ways that I conceptualize different ways of thinking about changes. As a result, I now make different distinctions between ways of thinking about exponential growth, and I have different priorities with respect to what students need to learn about exponential growth. As a result of making these distinctions, even the way in which I interpret literature that I have previously read is different.

In order to describe the subtle distinctions in my own thinking that have occurred as a result of this study, I must divide the story into three parts. The first part is the story of my ways of thinking before this study. It describes the interpretations of the literature that necessitated this teaching experiment, the goals that I held for the teaching experiment, and my design of the teaching experiment based on those old interpretations and goals.

The second part of this story is the story of the students and myself. It describes my interactions with two students across fifteen teaching episodes, as well the models of their mathematics that I have formed to explain their actions. The models of the students' mathematics that I formed of the students then become the object of discussion in the third part of this work.

In this third part, I discuss how the process of forming these models of the students' mathematics changed my own understanding of the relationships between variation, change, rate of change, and exponential growth. I place these old and new understandings in direct contrast with each other by reinterpreting the previously read literature from my new perspective, and by discussing the consequences of this work for teaching and learning exponential growth.

Thus this work addresses three questions: What are some ways in which students actually operationalize covariation? What are the consequences of those ways of resoning covariationally for the student's subsequent understandings of exponential growth (and for dynamical systems as a whole)? And what understandings of exponential growth should we set as goals for our students?

CHAPTER 2

STUDIES OF COVARIATION AND THE VERHULST MODEL PRIOR TO THE DESIGN OF THE TEACHING EXPERIMENT

In preparation for the design of the teaching experiment, I studied previous work on covariation, exponential growth, mathematical modeling, and the Verhulst model. The remainder of this chapter is the product of those studies, which are indicative of the thinking and understanding in which I based the teaching experiment. This work has been edited retrospectively in order to make clearer the contrasts between my thinking at the time of my experiment and my thinking after the experiment.

A Brief History of Covariation

In her doctoral dissertation, Rizzuti (1991) introduces covariation by providing a brief history of the development and formalization of the definition of function among mathematicians. She argues that the classical meaning of function involves expressing a relationship between varying quantities, and that through various proposals by a number of mathematicians, this meaning of function has been formalized into the modern set-theoretic mapping definition.

The meaning of covariation among mathematics educators is undergoing a similar process. Initially, the meaning of "covariation" relied greatly on intuition, but over time, mathematics educators have proposed increasingly formal refinements to these meanings, and the final choice of a definition for covariation is far from settled.

Finding an origin for the use of covariation in mathematics education literature is difficult. Search engines are of little use because "covariation" is much more commonly used as a statistical term, and statistics plays a large role in both mathematics and education literature. The approach that I settled on to investigate the development of meanings of covariation was to select a few modern articles on the subject, and follow the trail of citations back to develop a history of covariation. What I discovered was two parallel developments of covariation as a formal concept over a twenty-year period: one based on the work of Jere Confrey (Confrey, 1988; Confrey & Smith, 1994; Rizzuti, 1991; Strom, 2008) and the other based on the work of Patrick Thompson (Saldanha & Thompson, 1998; Thompson, 1988, 2008a, 2008b; Thompson & Thompson, 1992).

This history begins when Confrey (1988) proposed a revision "in the presentation of variables and functions in order to focus more on difference and change in the dependent variable (y1->y2) rather than on the input-output (x produces y) perspective of our traditional f(x) notation" (p 252). Confrey went on to speak about looking at the change in the dependent variable as the independent variable varies. Although this language suggests an already formed covariational mindset, this covariational language is not accompanied by the word "covariation" or references relevant to covariation. So, for the purposes of following the formalization and development of meaning of covariation, this is a starting point. Confrey was already thinking about covariation in 1988, but she hadn't put a name to it yet.

In the same volume, Thompson (1988) described a cognitive objective for quantitative reasoning. This cognitive objective included understanding a quantity as being the measurable quality of something. That is, a quantity is composed of a measurable property of something, and a magnitude that is the quantity's measure in some unit. Quantitiative reasoning, according to Thompson was “to reason about quantities, their magnitudes, and their relationships to other quantities” (p. 164). This quantitative reasoning becomes the basis for Thompsons’ covariational reasoning in later work (Thompson, 2008b).

To contrast the two beginning points, Confrey (1988) focused on successive values of variables, while Thompson (1988) focused on the measurement of properties of objects.

My first observed use of the word covariation is from 1991, in Rizzuti’s dissertation under Confrey. Here, Rizzuti took a very intuitive approach. She called upon the classical meaning of function to describe what she meant by covariation, but does not provide any formal definition. Some measure of familiarity with the term is assumed: “The ‘classical’ definition expresses a relationship between varying quantities, a covariation relationship” (p.24). Rizzuti’s description of the classical meaning of function gives insight into Rizzuti’s meaning of covariation as a student’s coordination of two continuously varying quantities, but she did not make this meaning explicit.

Neither of these articles had provided a formalization of covariation, a definition, or a description of the mental operations that characterize covariational reasoning. Continuing the work of Thompson (1988), Thompson and Thompson (1992) developed a theory of mental operations that compose an image of constant rate of change. The authors began a process of formalizing covariation when Thompson and Thompson developed four stages -- or levels -- of images of rate and ratio. A student at the first level would interpret two quantities in a ratio as constants. Covariational reasoning began at the second level: an internalized rate involved an iteratively additive process, in which a student visualizes the accumulation of one quantity in fixed amounts and makes a correspondence to accumulation of another quantity in fixed amounts. An interiorized ratio involved the same mental actions, but with the student anticipatimg that the ratio of the accumulations (the quotient) would remain constant. A fourth level of *rate* would be achieved when the student's image of accumulations becomes both simultaneous and continuous, and the reasoning is multiplicative: the student can reason that if one quantity changes by a factor of $\frac{m}{n}$, the other quantity does as well.

Unlike Rizzuti's (1991) description of covariation as continuous, Confrey and Smith (1994) advocated a discrete approach that focuses on changes between successive values of two variables. Confrey and Smith identified two types of constant changes, multiplicative change and additive change, and what they called covariational reasoning is a process of coordinating these types of changes in an iterative manner. This reasoning process could be represented with a system of difference equations, for example:

$$\begin{aligned} x_{m+1} &= x_m + c \\ y_{m+1} &= c y_m \end{aligned} \tag{1}$$

In response to Confrey and Smith, Saldanha and Thompson (1998) proposed a meaning of continuous covariation:

> An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. In the case of continuous covariation, one understands that if either quantity has different values

> at different times, it changed from one to another by assuming all intermediate values (Saldanha & Thompson, 1998, p. 2).

Saldanha and Thompson's meaning of covariation is ambiguous, however, in that it does not describe how one reasons about "assuming all intermediate values." One framework for describing reasoning about intermediate values can be found in Carlson, Jacobs, Coe, Larsen, and Hsu (2002).

Carlson et. al. (Carlson, et al., 2002) proposed five levels of covariation reasoning, and five mental actions that characterize these levels. The five mental actions that characterized these levels were (in order): "Coordinating the value of one variable with changes in the other," "coordinating the direction of change of one variable with changes in the other variable," "coordinating the amount of change of one variable with changes in the other variable," coordinating the average rate-of-change of the function with uniform increments of change in the input variable," and "coordinating instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function," (p. 357).

Mental action 3, "coordinating the amount of change of one variable with changes in the other variable" (p. 357), can be seen as analogous to Confrey and Smith's (1994) model of covariation as simultaneous difference equations. Mental Action 4, "Coordinating average rate-of-change of the function with uniform increments of change in the input variable" (p. 357), would address the issue of intermediate values if a student used those average rates-of-change to approximate a function with a piecewise linear function: one of the behaviors that Carlson et. al. associated with Mental Action 4 is "constructing contiguous secant lines for the domain" (p. 357).

Carlson et. al. also address intermediate values with Mental Action 5, "coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function" (p. 357); however, because the mental actions are used primarily as an assessment tool rather than a teaching tool, the authors do not describe a process of reasoning by which a student who uses Mental Action 4 might come to use Mental Action 5. So it is not clear what method of reasoning about intermediate values they attribute to level 5 students. One possibility is that the student imagines a limiting process of piecewise linear approximations, but other ways of reasoning are possible.

Breaking out of the chronological order of our studies for a moment, Confrey and Smith (1995) also proposed a way of thinking about intermediate values. In the construction of sequences by repeated addition or repeated multiplication the student would have to construct an operation for finding the value of the sequences at fractional indexes $1/n$ such that that when this fractional successor operation is composed with itself n times, the effect would be identical to the unit successor operation (of multiplying or adding a constant value) (Strom, 2008). Confrey and Smith (1995) proposed that intermediate values for sequences constructed by repeated addition could be found by arithmetic mean, while intermediate values for sequences constructed by repeated multiplication could be found by the geometric mean. This process is only briefly mentioned by Confrey and Smith, and a more detailed treatment of partial successors for geometric sequences is described by Strom (2008).

Another way of reasoning about Saldanha and Thompson's meaning of continuous covariation can be found in Thompson's (Thompson, 2008b) recent work. He characterizes an understanding of continuous variation as when student sees a quantity as having different values at different times, and he or she imagines that the quantity has those values at the beginning and end of an interval of conceptual time, and that the quantity assumes all intermediate values over that interval of time. The way that Thompson proposes thinking about these intermediate values over this interval of conceptual time is that they can then be broken up into sub-intervals, and at the endpoints of each sub-interval, the quantity has different values at different times, and the quantity assumes all intermediate values over that sub-interval of time. Over the course of the sub-interval of time, the quantity assumes all intermediate values, and in examining those intermediate values, the process becomes recursive. Covariation, in Thompson's (Thompson, 2008b) meaning of the word, occurs when a second quantity is seen to be varying over the same intervals of conceptual time. The relationship between these quantities comes from from coordinating the values at the endpoints of the intervals, sub-intervals, and sub-sub intervals that drive continuous variation.

In order to make a case that there is no one meaning of covariation commonly agreed upon by the mathematics education community, I have provided examples of meanings of covariation from several authors in a historical overview. The meaning of covariation is, thus, something that is still being negotiated by the mathematics education community. The following section will highlight some of the differences among those meanings.

Covariation and Exponential Growth

Both Thompson (Thompson, 2008a) and Confrey and Smith (Confrey & Smith, 1994, 1995) have described how a student, thinking covariationally in they way that they describe, might imagine exponential growth.

The Exponential in Confrey and Smith

Confrey and Smith (Confrey & Smith, 1994, 1995) connect their meaning of covariation to exponential reasoning with an operation called *splitting*. A split is an action by which “an object is being replaced by some fixed number of copies as a repeatable process” (Confrey & Smith, 1994, p. 148). Confrey & Smith (1994) also distinguish splitting from sharing – an action where the results are equal amounts, rather than equal copies.Splitting is distinguished from a second type of operation called *counting*. The distinction is most easily seen in situations where the actions are repeated. A repeated *n-split* results in iterated multiplications of an initial value by n. A repeated count of c would involve iterated additions of c. Exponential reasoning, according to Confrey and Smith, arises when students coordinate a repeated splitting action and a repeated counting action, as in the equations below:

$$\begin{aligned} x_{m+1} &= x_m + c \\ y_{m+1} &= n \cdot y_m \end{aligned} \tag{2}$$

The result of this coordination is what I would classify as a geometric growth function. With the tools of repeated count action and repeated splitting action, a student can only generate values of y on the domain $x = ck, k \in \mathbb{Z}$. Determining values of y

corresponding to rational valued x requires additional thinking machinery (Confrey & Smith, 1995; Strom, 2008). Specifically, it requires the ideas of composing and partitioning beyond Confrey and Smith's counting and splitting actions.

The act of composing splitting or counting actions occurs when a student imagines an action being repeated a certain number of times, and then imagines the equivalent single action that would generate the same result. For example, performing 3 splits repeated four times always generates the same final result as a performing a 3^4 split one time.

Partitioning a repeated splitting or counting action, described briefly by Confrey and Smith (1995) and touched on in greater detail by Strom (2008), occurs when a student imagines a single splitting or counting action, and then imagines the equivalent repeated action that would generate the same result after a certain number of repetitions. In the case of partitioning splitting actions, this reasoning results in a way of thinking that Strom (2008) refers to as "partial factors." Reasoning in partial factors is to assign meaning to rational exponents by the repeated multiplication meaning of exponentiation. That is, $3^{1/3}$ is the number such that, $3^{1/3} \cdot 3^{1/3} \cdot 3^{1/3} = 3^1$. So in this example, performing a single 3 split action is equivalent to a performing a $3^{1/3}$ split action 3 times.

This "partial factors" reasoning hinges on the idea that an n split is always equivalent to an $n^{1/a}$ split repeated a times. However, when modeling context is taken into account, this idea breaks down, because the domains generated by repeating the two different splits are different.

An example of the usefulness of this distinction is in the growth of insect populations. Insects all lay eggs at more or less the same time. The adults are killed during the winter, and the eggs hatch when the weather is suitable. A population that grows in this way is best modeled discretely for two reasons. First, if measuring the population of adults, the population fluctuates seasonally, reaching 0 in the winter, so intermediate values are discontinuous. Secondly, because all the insects lay their eggs at approximately the same time, the growth in the population is not distributed over an interval, but rather occurs all at once. In fact, between hatchings, the population declines due to starvation or predation, drops abruptly to 0 during the winter, and does not actually increase until the next hatching occurs.

If we imagine that the insects lay 900 eggs each year before their deaths, and that 4 of those eggs survive to become adults, then the population grows by 4 split each year. However we cannot say that the population doubles every six months in this context, because over the course of the year, the population is declining from the 900 eggs each insect lays to the 4 adults of the next generation that actually survive. An "equivalent" 2-split every half-year predicts growth that does not occur.

Another key feature of exponential reasoning is that the rate of change of the function is proportional to the value of the function. This feature is critical to the use of exponential functions in dynamical systems modeling. Using Confrey and Smith's (1994) meaning of rate, however, this statement is not true. Confrey and Smith's meanings of ratio and rate are based on their meaning of unit, which is defined by Confrey and Smith as "the invariant relationship between a successor and its predecessor;

the unit is created as a result of repeated action," (1994, p. 142). Thus equation $x_{m+1} = x_m + c$ has an additive unit *c*, and the equation $y_{m+1} = cy_m$ has a multiplicative unit *c*. This multiplicative unit *c*, is also called a *ratio*. Based on this meaning of unit, Confrey and Smith (1994) define a *rate* as "a unit per unit comparison", and Confrey and Smith (1995, p. 75) distinguish between an additive rate ("the difference per unit time") and a multiplicative rate ("the ratio per unit time").

The repeated actions in exponential reasoning as constructed by Confrey and Smith are an *n*-split, and a *c*-count. The rate is then a comparison of the multiplicative change *n* to the additive change *c*, rather than a ratio of the additive changes $\Delta y/\Delta x$. So an exponential function has a constant rate of change, not a rate that is proportional to the value of the function.

Thompson's Construction of the Exponential

While Confrey and Smith's exponential is based on the idea of a constant multiplicative rate based on the coordination of a repeated splitting action and a repeated counting action (Confrey & Smith, 1994), Thompson's exponential is based on the idea of a rate as the multiplicative comparison between two continuously changing quantities (Thompson, 2008a). Thompson's meaning of *rate of change*, in contrast, is based on coordinating the variation of quantities in time. When two quantities change in time[1], there is the possibility that the multiplicative comparison (*ratio*) between these quantities measured at any frozen moment in time is always constant. When this constant ratio is

[1] Time is also a quantity that changes in time. It changes by the identity function.

reflectively abstracted to be independent of the frozen moments of time at which these quantities are measured, one has constructed a constant *rate* (Thompson, 1994b).

A Thompson rate differs from the Confery and Smith rate in one primary way: a Thompson rate is not iterative, but rather proportional. To use speed as an example, a speed of 5 miles per hour does not mean every hour a person walks five miles, but rather that for any time *t*, the number of miles *d* is five times the number of hours, and that in any fraction or multiple of that time, the person traveled an equal fraction or multiple of the distance (Thompson & Thompson, 1994).

This conception of rate differs from iteration in that conceiving of fractions of distance or fractions of time of any size requires reasoning on the continuous and dense number system that Thompson covariation is based in. The relationship must hold not only for every iteration, but also for every possible size of an iteration step. The result is a notion of rate that is very similar to uniform density: any amount of volume, selected anywhere, has a corresponding amount of mass in fixed proportion. Any amount of time, selected from any starting time, has a corresponding amount of distance, in fixed proportion.

Thompson (Thompson, 2008a) develops exponential reasoning in the context of simple interest and compound interest. In simple interest, the amount paid per unit time is a constant proportion of the principal. If this is generalized as a Thompson rate, then the amount paid in interest is proportional to both the initial investment and the time. Half the initial investment means earning half the interest in the same amount of time, a seventh of the time means a earning a seventh of the interest for the same initial investment.

In this simple interest model, the principal is set at the beginning of the investment, meaning that the total amount of the investment over time can be represented as a constant linear function. The equation below demonstrates this relationship with a fixed principal of $1000, and a fixed interest accumulation rate of 1.08*1000 dollars/year.

$$y = 0.08(1000)x + 1000 = 1.08(1000)x \tag{3}$$

Now, imagine that that interest is compounded at the end of every year. Although the interest "rate" of 8% stays the same, the rate of accumulation of the number of dollars per year changes at the end of every year, because the principal investment changes at the end of every year. Principal, then becomes a step function of time, and the amount of the investment is a piecewise linear function, where the rate of change increases every year. At the end of each compounding period, the total accumulated investment becomes the new principal, so the principal and the investment are equal at the end of every compounding period.

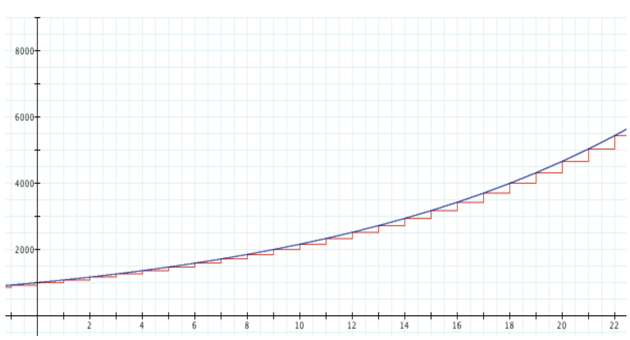


Figure 1. An investment compounded annually at 8% per year. The blue function, showing account balance over time, is a piecewise function consisting of line segments.

The red function, showing the value of the principal over time, is a geometric step function.

The (Thompson) rate of the accumulation of the number of dollars with respect to time (call this rate *a*) is now a function of a changing principal:

$$a(p) = 0.08\,p(x) \tag{4}$$

This rate is constant for any subinterval of the compounding period. Over the course of a year, the investment changes by *a*(*p*). Over a seventh of a year, the investment changes by *a*(*p*)/7.

So the function for the value of a compound interest account is a function created by concatenating simple linear interest accounts over each compounding period. The function for this compound interest account, of 8% of the principal investment per year, beginning with \$1000, and compounded *n* times per year becomes:

$$y = 0.08\left[1000\left(1+\frac{.08}{n}\right)^{\lfloor nx \rfloor}\right]\left(x-\frac{\lfloor nx \rfloor}{n}\right)+\left[1000\left(1+\frac{.08}{n}\right)^{\lfloor nx \rfloor}\right] \tag{5}$$

When compared with the equation for simple interest (3) above, we see that the origin of this compound interest function (5) is a simple interest account beginning with an investment of $\$\left[1000\left(1+\frac{.08}{n}\right)^{\lfloor nx \rfloor}\right]$ at time $\frac{\lfloor nx \rfloor}{n}$, instead of beginning with \$1000 at time 0. The $\left[1000\left(1+\frac{.08}{n}\right)^{\lfloor nx \rfloor}\right]$ is the traditional formula for the value of the principal account, found by repeated applications of the distributive property as shown in Equation 3.

Now compound the investment more frequently, over minutes, seconds, and nanoseconds. As this occurs, the interval of time that a particular rate of change of principal with respect to time is in effect becomes smaller and smaller, and the rate changes more and more frequently. This results in the account balance function smoothing out from piecewise linear to an exponential curve.

This also generates the exponential property that the rate of change of the function is proportional to the value of the function. Over every interval the principal accumulates at a constant rate, and that rate is proportional to the principal at the beginning of the interval. At the same time, the intervals at which the investment and the principal are equal also occur more and more frequently. In the limit, the investment is always equal to the principal, and the rate is no longer a constant Thompson rate $a(p)$, but rather the limit of a rate over smaller and smaller intervals ($\dot{y}$). But the proportional relationship between $a(p)$ (which becomes $\dot{y}$) and the principal never changes, leading to the equation:

$$\dot{y} = 0.08\,p(x) = 0.08\,y \tag{6}$$

The Verhulst Model

Smith, Haarer and Confrey (1997) presented an account of a collaboration between a mathematical epidemiologist, “Carlos,” and an evolutionary biologist, “Sara.” The purpose of the collaboration was to teach a graduate-level class on mathematical population biology. During the course of the class, students collaborated on group projects involving the interactions of multiple species. Based on their observation of the course, the authors described vast differences in the participants' ideas of the meaning and purpose of mathematical modeling. These differences existed both between the instructors, between students, and between students and instructors. In the course of describing the different perspectives among students and instructors, Smith, Haarer and Confrey (1997) made use of one subject's (Carlos) terminology to make sense of these differences in meaning. These terms: *descriptive, theoretical, ad hoc, and first principles*, are useful for distinguishing between different ways of understanding the Verhulst model as well.

The subject of the study, Carlos, distinguished between *descriptive* and *theoretical* models, and between *ad hoc* and *first principles* models. A descriptive model attempts to capture everything about a system in order to duplicate reality as closely as possible and make accurate predictions. A theoretical model, in contrast, discards everything about a situation except what is relevant to a specific question of interest. The difference then, is in the goal of the modeler: to incorporate as much as possible, or to discard as much as possible.

The second distinction is about the way a model is constructed. An ad hoc model is based on observation, and the modeler develops formulae by techniques such as curve fitting. First principles models, in contrast, are built from a non-mathematical understanding of the system. The modeler develops formulae by imagining the interactions of objects in the situation and the consequent effects on measurable quantities.

The Verhulst model is not a very good model for population growth. As a model, it is overly simplistic. The assumptions inherent in the model are questionable at best, outright inaccurate in other cases, and the predictions that the Verhulst model makes about populations do not stand up to scrutiny (Kingsland, 1982). Fundamentally, the Verhulst model is an ad-hoc model – an equation chose to fit data rather than an equation chosen based on a mathematical understanding of the mechanics of population growth (Smith, et al., 1997; Verhulst, 1977). Attempts to interpret the Verhulst model in terms of biological principles end in failure (Gabriel, Saucy, & Bersier, 2005).

Nonetheless, the Verhulst model remains one of the most popular models in population biology, used in the instruction of both ecology and mathematical biology (Bergon, et al., 1996; Brauer & Castillo-Chavez, 2000; Edelstein-Keshet, 1988). The simplicity of the model makes both the mathematics and the biology of it accessible in ways that "better" models are not. Partially despite, and partially because of its flaws, the Verhulst model and its variants are an integral part of the culture of mathematical biologists, and it serves as a starting point for inspiring a large body of work in these fields (Kingsland, 1982).

Derivations of the Verhulst Model

Using the exponential family of functions to measure population growth is named for Thomas Malthus, who observed that populations had a tendency to grow geometrically, and predicted that population would potentially exceed food supply in the future (Malthus, 1826). Both geometric and exponential growth models are called the

Malthus model. This study deals primarily with the continuous exponential growth model:

$$\frac{dN}{dt} = rN \tag{7}$$

where N represents the number of organisms (population), t is time measured in some unit, and r is the per-capita rate of growth of the population with respect to time, sometimes called the Malthusian growth rate, or the exponential growth rate. Although the value of N is dependent on time, conventional notation in population biology is to represent the measurement of something (population at a particular time) as a variable, N, rather than as a function, $N(t)$.

The Verhulst model was initially derived independently by Pierre-François Verhulst in 1838, and by Pearl and Reed in 1920. In both cases, the authors were attempting to improve on the continuous Malthus model of exponential growth by proposing a maximum limit to the population. In both cases, the authors took an ad-hoc descriptive approach, hoping to fit population data to a curve, and to use that curve to make predictions.

Verhulst (1977) took an approach of beginning with the Malthus model in differential equation form, and subtracting an unknown function of the population. In his notation:

$$\frac{dp}{dt} = mp - \phi(p) \tag{8}$$

where p represents a population, t represents time, and m represents the Malthusian (or exponential) growth rate. In order to determine the definition of φ, Verhulst considered four possible functions, $\phi(p) = np^2$, $\phi(p) = np^3$, $\phi(p) = np^4$, $\phi(p) = n \ln p$ and used curve fitting with population data from Paris to select $\phi(p) = np^2$ as the best fit to his data.

Pearl and Reed (1977) took an approach of studying the function $p(t)$, rather than the differential equation form. By specifying a number of properties that the function would need to have, such as horizontal asymptotes, a point of inflection, and concavity, Pearl and Reed chose the function

$$p = \frac{be^{at}}{1 + ce^{at}} \tag{9}$$

and described methods of fitting this curve to population data. Pearl and Reed did not print a differential equation form of the model in their article.

An example of a re-interpretation of the Verhulst model to derive the model from biological principles comes from Edelstein-Keshet (1988). Edelstein-Keshet began with a Malthus model, and altered it by assuming that the Malthusian growth rate growth of a population N would be proportional to the availability of a nutrient, and that the amount of that nutrient, C, would be reduced by population growth. Edelstein-Keshet represents these relationships with a system of differential equations:

$$\begin{aligned} \frac{dN}{dt} &= kCN \\ \frac{dC}{dt} &= -\alpha \frac{dN}{dt} = -\alpha kCN \end{aligned} \tag{10}$$

where α represents the amount of nutrient consumed in order to produce one new unit of population. Solving for $C(t)$ in terms of N yields another form of the Verhulst model:

$$C(t) = C_0 - \alpha N(t)$$
$$\frac{dN}{dt} = k(C_0 - \alpha N)N \tag{11}$$

In this form, it can be seen that the amount of nutrient C decreases from an initial value C_0 at a constant rate with respect to the population N. As the population N increases with time t, the amount of nutrient decreases, and the Malthusian growth rate of the population decreases. The population N at time t can be found without knowing the amount of nutrient available at time t, so only the second equation is necessary to specify the behavior of the population.

Another common approach to deriving the Verhulst model is based closely on the property that the rate of change of an exponential is proportional to its value. This approach begins by looking at per capita growth rate (Bergon, et al., 1996; Gabriel, et al., 2005). The Malthus model assumes a constant per-capita growth rate, meaning that the contribution that an individual makes to the growing population is always constant[2], meaning that an individual produces new offspring continuously at a constant rate:

$$\frac{1}{N}\frac{dN}{dt} = r \tag{12}$$

This derivation of the Verhulst model begins by assuming that rather than making a constant contribution, the per-capita growth rate decreases linearly with population pressure until it reaches a maximum sustainable population of K (Figure 2).

$$\frac{1}{N}\frac{dN}{dt} = r(1 - \frac{N}{K}) \tag{13}$$

[2] To make an analogy to a discrete system, all mothers give birth to the same number of babies at the same constant interval.

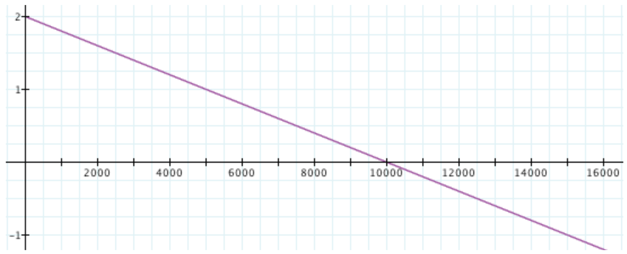


Figure 2. Per-capita rate of change as a function of population with r=2 and K=10000.

This generates the most common ecological form (Gabirel, Saucy, Bersier, 2005) of the Verhulst model"

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) \tag{14}$$

where *r* represents the Malthus exponential growth rate, and *K* represents the population's carrying capacity – the maximum sustainable population. A population in excess of *K* will die off, resulting in population decrease.

Although all four derivations result in different forms of the Verhulst model, with different contexts, different assumptions, and different parameters, a little bit of algebra will show that all four forms are mathematically equivalent.

Learning the Verhulst Model

Research on students learning the Verhulst model is scanty. Searches for combinations of "learning," "teaching," with "Verhulst" or "Verhulst model" in the ERIC, JSTOR, and SprinerLink database produced only one hit, by Harding (1993), who, in the review of a textbook, asserted that the Verhulst model was accessible to fifth-form students – approximately equivalent to U.S. freshman or sophomore college students. Searching for the same combinations using the more common, but less precise term "logistic model," produced larger numbers of hits. The term "logistic model," however, is ambiguous. Most of these articles involved the statistical logistic regression model. Other articles dealt with a logistic model in economics. A smaller number of articles involved the logistic difference equation. The smallest number of articles involved the logistic differential equation.

Iovinelli (1997) developed a lesson plan on using the analytical solution of the logistic differential equation (the Pearl and Reed form) of the model to fit data. Iovinelli's lesson plan is very similar to the Pearl and Reed derivation in that it establishes behaviors that the curve should have, and then presents an equation for the curve without any biological justification. The lesson plan is based purely in ad-hoc descriptive modeling.

Jones (1997) described a classroom experiment in which undergraduate students working with MAPLE were asked to fit data to the Malthus and Verhulst models. The students used MAPLE to find an explicit solution to the Verhulst model, but the experiment never covered derivations or interpretations of the model, nor did Jones discuss prerequisite mathematical understandings. Berry described Jones' data fitting problem as one that "does not develop theoretical modeling skills" (2002, p. 214).

Ang (2004) describes a modeling process in which the Verhulst model was fit to data, and then modified for a better fit. The intention of the paper is to propose that the modeling process be used as a guided activity for students to engage in genuine applied mathematics research. Ang's modeling process involves both the differential equation form of the Verhulst model and the analytical solution, and both ad hoc and biological justifications for modifications that are made to the logistic model. It does not, however, describe how the Verhulst model might be justified to students in the first place, or what understandings of biology or of mathematics are necessary in order to either derive or critique the Verhulst model.

Borba and Villareal (2005) described two student research projects that developed during undergraduate biology classes that emphasized modeling and technology. For the first project, in 1999, the students plotted data relating the rate of photosynthesis in chloroplasts to the intensity of light, resulting in a ‘s’ shaped curve. The teacher informed the students that the curve was called a ‘logistic’ curve, and pointed the students to resources in a book. For the second project, in 2001, the students were concerned with fitting Bovine Spongiform Encephalopathy data. Initially the students attempted to fit annual cases to a 6^{th} degree polynomial. Following upon a suggestion from the teacher, the students instead chose to fit cumulative cases to a logistic curve. In both studies, the authors described the process of fitting the data, but state very little about the students’ mathematical or biological understandings of the logistic model, or the different modeling perspectives assumed by each choice.

Lingefjärd (2006) uses a common variation of the Verhulst model (the logistic model with constant yield harvesting) as an example of a model that could be used in modeling education, but does not provide any description of how it might be used, or how students would need to think about it.

All of the above studies fall into one of two categories: either they describe an experiment in which students are given the logistic model and asked to fit data to it (Ang, 2004; Borba & Villareal, 2005; Iovinelli, 1997; Jones, 1997) or they assert that the logistic model could be used for teaching mathematical modeling, but provide no details as to how (Berry, 2002; Harding, 1993; Lingefjärd, 2006). I have been unable to find any literature that discusses how students might develop an understanding of either the mathematics or the biology of the Verhulst model, and certainly there has yet to be any work developing the Verhulst model from covariational reasoning.

Covariation and the Verhulst Model

There has been, to my knowledge, no work published on how operations of covariation – that is different "ways of thinking" about two variables or quantities changing simultaneously – play a role in the derivation of the Verhulst model or in the study of its behavior. However two papers describing different ways of covariational reasoning (Carlson, et al., 2002; Confrey & Smith, 1995) also hint at covariation playing a role in understanding differential equations, and this seems to be intuitively correct. It makes sense that the way in which a student imagines variables or quantities changing might impact their understanding of dynamical systems, of which differential equations are a sub-type. However, these papers do not discuss possible connections between the authors' models of covariational reasoning and differential equations in any depth.

Carlson et. al. (2002) did not make any explicit connections between covariation and differential equations. The article discusses covariational reasoning as a way for students to better understand functions, and cite Rasmussen (2001) –reviewed below -- as evidence that function understanding is important for differential equations understanding.

Confrey and Smith (1995) stated that differential equations are built by modeling and coordinating additive rates of change in two variables. The authors provide no details about how someone reasoning covariationally would think about deriving a differential equation, and do not address how a student might think about the most important characteristic of a differential equation: that the rate of change of the dependent variable with respect to the independent variable is not a function of the independent variable, but rather a function of the dependent variable.

I performed JSTOR, ERIC, SpringerLink, and Google scholar searches for articles on “differential equations” and “covariation.” The vast majority of the articles found by these searches were on the topic of stochastic differential equations, in which covariation had a statistical rather than an educational meaning. After eliminating these results, only three papers remained: Confrey and Smith (1995), Carlson, Jacobs, Coe, Larsen, and Hsu (2002), and Trigueros (2008). Of these results, only Trigueros was new.

Trigueros (2008) engaged in a classroom experiment in differential dynamical systems based on model-eliciting activities and APOS theory. The student modeling process that she described indicates that students initially spent a great deal of time identifying appropriate quantities to base the model on. Trigueros referred to this perspective as a “covariation” perspective and contrasted it with students’ later phase plane examination of their chosen differential equation. Although Trigueros concluded that this type of quantity-based reasoning serves as a basis for differential equation modeling, she did not examine the operations of covariational change or how differential equations might be derived and studied from a covariational perspective.

My second method of finding literature was to look at the citations of Carlson, Jacobs, Coe, Larsen, and Hsu's (2002) mention of differential equations, and follow the work of two authors: Zandieh and Rasmussen.

Zandieh (2000) described a framework for assessing and categorizing students' understanding of derivatives, similar to Carlson, Jacobs, Coe, Larsen, and Hsu's (2002) framework for assessing and categorizing students' understanding of covariation. Also like Carlson et. al., Zandieh's assessment framework does not describe how students might learn the ideas being assessed.

Rasmussen (2001) wrote that the idea of a solution to a differential equation requires a different meaning of solution for students: they need to think of a solution as a function, rather than as a number. Rasmussen also discussed a second type of solution, which he termed an "equilibrium solution," in which students study equilibria as a method of examining classes of behavior of the system. This distinction between types of solutions parallels the earlier distinction made between descriptive modeling (for which a function solution is more useful) and theoretical modeling (for which a qualitative behavioral solution is more useful).

Based on these views of "solution", Rasmussen (2001) highlights the importance of the student's understanding of function and quantity, as well as the students' images of stability and numerical approximation. Rasmusssen does not, however, make connections between these issues and the operations of covariational reasoning.

Synthesis: Choosing a Framework for the Teaching Experiment

None of the studies in learning the Verhsult model (Ang, 2004; Berry, 2002; Borba & Villareal, 2005; Iovinelli, 1997; Jones, 1997) discussed how the Verhulst model might be derived from covariational reasoning, however, the “modern” first principles derivations of the Verhulst model used to teach in ecology and mathematical ecology (Bergon, et al., 1996; Brauer & Castillo-Chavez, 2000; Edelstein-Keshet, 1988) rely two ideas: that the Verhulst model is a modification of the Malthus model of exponential population growth, and that the derivation of the model relies on modifying the differential equation form of the Malthus model. That is, the derivation of the Verhulst model relies on understanding and modifying the Malthus growth property that the rate of growth of the population is proportional to the value of the population.

Because of this fact, I chose to base the instruction design of my teaching experiment on Thompson’s covariation and Thompson’s construction of the exponential, which more easily address the idea of rate proportional to amount (Thompson, et al., 2008).

Choosing a Derivation of the Verhulst Model

The two “modern” approaches to deriving the Verhulst model (Bergon, et al., 1996; Edelstein-Keshet, 1988) both rely on differential equation models, but are actually quite different in terms of the ways of thinking involved. Edelstein-Keshet’s approach is more involved both biologically and mathematically. By modeling the impact of the population on food supply, Edelstein-Keshet provides a first-principles justification for quadratic nature of the differential equation form of the Verhulst model. However, part of that first principles justification requires finding the analytic solution to the differential equation describing food levels. Bergon et. al.'s derivation is less biologically and mathematically sophisticated, in that it does not require solving a differential equation and it does not biologically justify the linear per capita rate of change model used to derive the Verhulst model.

Although a student following the Edelstein-Keshet approach has the advantage of building a much stronger image of the biological situation that the Verhulst equation models, the Bergon et. al. approach has the advantage that it does not require as much facility with manipulating differential equations. With a target audience of high school students in mind, this was a concern in selecting a derivation to teach, and ultimately I chose to base my instruction on the Bergon et. al. derivation.

Population Rate of Change and Per-Capita Rate of Change

The per-capita growth rate derivation of the Verhulst model (Bergon, et al., 1996; Gabriel, et al., 2005) relies on the proportional relationship between exponential rate of change and exponential function value but introduces a slightly different concept: per-capita rate of change. Two formulations for the Malthus model are below:

$$\frac{dN}{dt} = rN \qquad\qquad \frac{\dot{N}}{N} = r$$

population rate of change   per-capita rate of change

The choice of notation above is deliberate. I am using the Leibniz notation for rate of change to highlight a multiplicative comparison between population (in the Malthus model) and time – this multiplicative comparison of population to time (or this *rate*, in the Thompson sense) is calculated as a function of the current population. In contrast, my choice of notation for per-capita rate of change is to highlight that in order to interpret per-capita rate of change, the rate of change must be seen as a single quantity with a single magnitude, rather than as a relationship between two quantities, and that the model is being interpreted as describing a multiplicative comparison between the quantities rate-of-change and population. This multiplicative comparison of rate to population is constant.

The rate of change of the population with respect to time does not have the property that as I vary time, the change in population is proportional to the change in time. Interpreting this rate of change as a *rate* in the Thompson (Thompson & Thompson, 1992; Thompson 1994a) sense of a constant proportional relationship between two quantities requires a little bit of work. At any moment in time, imagine a hypothetical situation in which the growth process suddenly stops counting (or for a biological interpretation, from that time forward, all offspring are sterile, while the parents continue to produce children); the rate of growth that will continue from that moment on is the Thompson rate corresponding to the rate of change. Interpreting the rate of change of the Malthus model as a Thompson rate means imagining that if, at any moment in time, the existing population continued to produce offspring, but all offspring were sterile, then the population would grow at a constant rate proportional to the size of the fertile population – (The population at the time when the sterility disaster occurred).

In contrast, the per-capita rate of change can be viewed as a Thompson rate without any need for hypothetical situations. As I vary the amount of the population (now the independent variable), the amount that the rate of change (now the independent variable) changes is a constant multiple of the change in population. This is the exponential property.

Per-capita rate of change can also be interpreted in a different way: as the equal partitioning of the rate of change among individuals. This partitioning highlights a previously unspoken assumption of the Malthus model: that all members of the population are reproductively indistinguishable. Because every individual is identical in a Malthus model, the contribution that each individual makes to the growth of the population is identical, and vice-versa. In the partitioning interpretation, per-capita rate of change describes a multiplicative comparison between the rate at which an individual contributes offspring to the population, and time. This multiplicative comparison can also be interpreted as a Thompson rate: as I vary time, the amount that any (single) individual contributes to the growth of the population is proportional to the amount of time that has passed.

A slightly over-simplistic continuous metaphor for the individual view of constant per-capita rate of change could be developed as follows. Instead of measuring the population in individuals, we imagine measuring the population in mass units the size of an adult individual. Then imagine a mammal that produces only one child at a time. From the time of conception to adulthood, the child grows at a constant rate. As soon as the child reaches adulthood and begins producing its own children, it stops growing, and the parent of the child begins a new child. Thus if we consider the contribution of the parent to be the mass of its first generation children, the parent is contributing child mass at a constant rate.

The reality of exponential growth is more complicated than this, however. From the multiplicative comparison meaning of per-capita rate of change, the rate-of-change of the mass of the population with respect to time is always proportional to the mass of the population. So the constant mass contribution metaphor only works if the child is also producing offspring at a rate proportional to its own mass. In other words, the child must be conceived pregnant. This does not change the rate at which the parent contributes mass, but is necessary in order to maintain the exponential property over the whole population.

Thus we distinguish between the growth that an individual contributes (growing at a constant per capita rate) and the growth that the population contributes (growing at a rate proportional to the population).

Deriving the Verhulst Model

Either interpretation of per-capita rate of change can be used to derive the Verhulst model. In both interpretations, the modeler must ask “What would happen the the per-capita rate of change was subject to population pressure?” But the quantitative interpretation of this question differs. From the rate of change/population version, the modeler must ask: What would happen if, instead of this constant relationship, as I vary the total population, the amount that the rate of change increases based on the change in population decreases based on the value of the total population? From the individual contribution/time version, the modeler must ask: What would happen if, instead of this constant relationship, as I vary time, the amount that every individual contributes to the growth of that population over that time period decreases based on the value of the total population?

However this question is formulated, modeling this question with a linear relationship between population and per-capita rate of change results in the per-capita growth rate derivation of the Verhulst model (Bergon, et al., 1996; Gabriel, et al., 2005). The choice of a linear relationship between per-capita rate of change and population is essentially arbitrary. In the Bergon et. al. derivation, there is no biological justification for using a linear relationship, only ease of computation.

However, the derivation can be made from a few assumptions. There exists a population level K at which the population stops growing, and everybody stops producing children. When the population is at its minimum stress (formulated as *P*=1 or *P*=ε), there is a known natural growth rate *r*; for simplicity, assuming that as the population grows from 0 to *K*, , the per-capita rate of change decreases from *r* to *K* linearly. Using the points (0,*r*) and (*K,* 0) to find the question of the per-capita growth line gives the following form of the Verhulst equation.

$$\frac{1}{N}\frac{dN}{dt} = r(1 - \frac{N}{K}) \tag{15}$$

Behavior of the Verhulst Model

Another key component of understanding the Verhulst model is the ability to describe and justify the behavior of the model. As a hypothesis, I propose that describing the behavior of the logistic differential equation can be accomplished by a student that uses both Thompson covariation, and an additional way of reasoning: the phase plane. By reasoning in arbitrarily small steps, the student uses the phase plane to perform a line of reasoning very similar to numerical analysis of an ordinary differential equation. By reasoning in small intervals, and imagining continuous variation over those intervals, the students can fill in the gaps between the steps and compensate for the inaccuracies of discretizing the model.

Imagine for a moment that we begin with a population $\mathrm{N}(t)$ between 0 and *K*/4 (at *t*=0). This population has a corresponding rate of change $f(\mathrm{N}(t)) = rN(t)(1 - N(t)$. If we fix this rate of change as a Thompson rate by imagining a line with this rate of change, and allow biological time take a small discrete step from *t* to *t*+ε, then this results in a slightly larger population approximated by $N(t+\varepsilon) \approx \varepsilon f(N(t)) + N(t)$.

A new population means a new, slightly larger rate of change, and the next increment in biological time results in a slightly larger accrual of population than before (Figure 3). This process continues until the population passes *K*/2, at which point it can be observed on the graph (Figure 3) and by this method that each accrual in population results in a slightly smaller rate of change. Between *K*/2 and K the population continues to increase, but each accrual of population is smaller than the last. These accruals can be plotted in a time series graph that approximates the solution to the logistic curve. Other starting points, such as $N(0)=0$, $N(0)=K$, $K/2<N(0)<K$ and $N(0)>K$ provide a complete picture of possible behaviors (Figure 4).

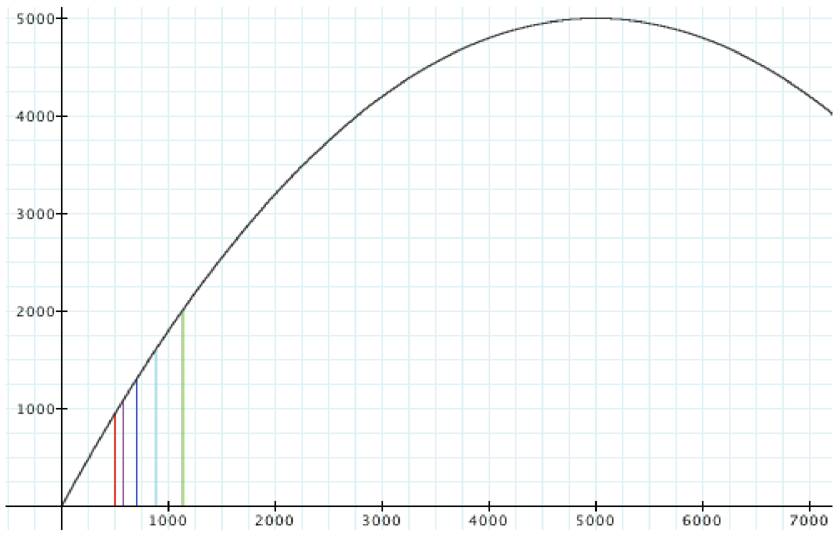


Figure 3. A phase plane diagram of the logistic model. The vertical axis measures rate of change. The horizontal axis measures population.

In Figure 3, the vertical bars measure the rate of change for specific populations. The rate of change measured by each bar (and the size of the covariational increment) specifies the change in population that determines the location of the next bar. As the rate increases, the bars fall farther and farther apart.

This discrete approach results in a rather serious error. No matter how small the step size is, the approximate population will always overshoot the stable equilibrium at $N(t) = K$, resulting in the model exhibiting damped oscillations around K, rather than a monotone approach. If a continuous approach based on Thompson covariation is used instead, this error does not occur. Rather than take discrete time steps, imagine that time varies continuously over all the values between the time steps t and $t+\varepsilon$. Then the approximate population N varies continuously over all values between $N(t)$ and $N(t+\varepsilon) \approx \varepsilon \dot{N}(N(t)) + N(t)$, and the rate of change varies continuously over all values between $\dot{N}(\mathrm{N}(t))$ and $\dot{N}(\mathrm{N}(t+\varepsilon))$. However, if 0 is in the interval $[\dot{N}(\mathrm{N}(t))$,

$\dot{N}(\mathrm{N}(t+\varepsilon))$], then in order for the rate of change to reach $\dot{N}(\mathrm{N}(t+\varepsilon))$, population must have at some point come to a stop and the entire process would have halted, meaning that N can never pass K.

This understanding of the logistic is distinct from a simpler analysis of the phase plane exemplified by thinking: positive [rate of change] means move [population] right, negative means move left. This simpler analysis does not generate the shape of the logistic curve, only its direction, unless "Postitive rate move right, negative move left" is a shortcut method that develops from having created a scheme that anticipates the behavior of one-dimensional ODEs after internalizing a process like the one outlined above.

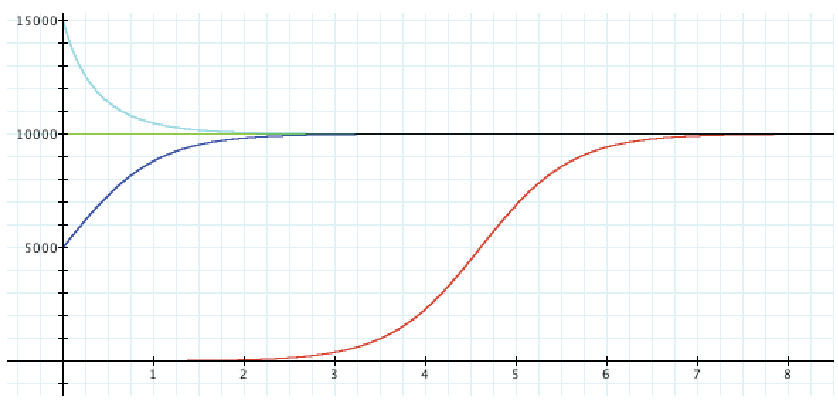


Figure 4. Behavior of the Verhulst model (r=2, K=10,000) for various initial conditions including N(0)=K (green) and N(0)=0 (x-axis).

Summary

This section provides a snapshot of my thinking prior to the teaching experiment, as I was preparing to design the experiment itself. Described here is my thinking about different understandings of covariation prior to the teaching experiment, my thinking about the interaction between different ways of thinking about covariation and exponential growth, and my thinking about how those interactions could be built upon by both teacher and student to construct a detailed understanding of the Verhulst model. It is here, at this point that I also began to set goals for the students who would be participating in my teaching experiment. I wanted them to derive the Verhulst model using a per-capita rate of change approach, and I wanted them to describe the behavior of the Verhulst model from the phase plane by using small linear approximations. Reaching these goals meant helping the students develop an understanding of exponential growth that was compatible with creating phase plane diagrams and interpreting them in arbitrarily small linear approximations. The design of the teaching experiment in the next section was based on these understandings and goals.

As the experiment itself began to play out, my own understandings and goals began to change, due to two primary factors: firstly, as the students struggled, it became more and more apparent to me that the way in which the students thought covariationally had an even greater impact than I originally anticipated; and secondly, my own understandings of the different ways of thinking about covariation changed, leading me to reinterpret the works of Thompson and Confrey and Smith very differently than I have here, in this snapshot of my initial thinking.

The remainder of this work is the story of the changes in my own thinking and the reasons for them, framed in the context of the story of two students learning about exponential growth.

CHAPTER 3

TEACHING EXPERIMENT DESIGN

This study is a part of a larger Teachers Promoting Change Collaboratively (TPCC) research project. Over a period of three years, the TPCC engaged in a classroom intervention, following students as they progressed as a class though Algebra I, Geometry, and Algebra II. At the time of this study, the class was three quarters of the way through Algebra II. This classroom provided an opportunity to study students who had already been exposed to covariational discourse, at a time in their mathematical development when exponential functions were part of the curriculum. However, the original purpose of this intervention was to provide video cases of teachers engaged in instruction to be used for teacher professional development, and the design of the intervention has been focused toward this purpose. Because the intervention in Algebra II was already underway, there were constraints on the design of this study.

The structure of the larger TPCC intervention involved a class, lead by the normal teacher of that class, and two to three researchers in the rear of the classroom observing each class. Occasionally a researcher would teach a class, however, the bulk of instruction came from “Liz”, the teacher of record. The researchers made observations and recommendations during the debriefing and planning sessions. The intervention also included regular debriefing sessions between the teacher and the researchers immediately after each class, a weekly planning session, and one-on-one interviews with the students. The class, debriefing sessions, and interviews were all videotaped, while the planning sessions were audio-recorded. Students' understandings were explored in the one-on-one interview sessions with the students, which were conducted by the researchers.

Teaching Experiment

The teaching experiment at the core of this study occurred approximately three quarters of the way through the school year, in March. Its overall design was based on the teaching experiment methodology of Steffe and Thompson (2000). The purpose of this teaching experiment was to create models of the students' understanding of covariation, exponential reasoning, and the Verhulst model in order to study how these components of the students' mathematics interact in their learning of exponential reasoning and differential equations modeling

Following the methodology of Steffe and Thompson, the teaching experiment consisted of a series of teaching episodes, formatted similarly to interviews already used in the intervention. Each teaching episode was designed to consist of myself as a teaching agent, the two students (nicknamed Derek and Tiffany), a witness to the teaching episodes (Pat Thompson), and various methods of recording what transpired in each episode – scans of student work and video recordings of the teaching episodes. However, due to the difficulties of coordinating the schedules among all who were involved, not all individuals were able to attend every teaching episode, making the only constants in each teaching episode myself, at least one student, and a video camera. The episodes that were observed were followed by debriefings sessions between the teaching agent (myself) and the witness (Pat). There were a total of fifteen teaching episodes and eight debriefing sessions.

The Exploratory Teaching Interview

The design of each teaching episode was based on the one-on-one interview methodology already in place, which we had come to refer to as an "exploratory teaching interview3" The exploratory teaching interview was a particularly informal version of an episode of a Steffe and Thompson teaching experiment, and this made it easy to adapt the exploratory teaching interview into a teaching experiment for this study. An exploratory teaching interview was always a one-on-one interview, and so did not implement the full Steffe and Thompson teaching experiment methodology. An exploratory teaching interview consisted only of a teaching agent, a single student, and a video camera to record the single teaching episode.

During an exploratory teaching interview (ETI), the interviewer asks the student to engage with a written mathematical problem, or a series of mathematical problems. The interviewer writes the text of these problems with the anticipation that the student will have difficulties with the problem, and that the nature of these difficulties will reveal information about how the student is thinking about the problem. The worst thing that can happen to an interviewer is that the student breezes right through the problem, which results in very little data and a strong temptation to attribute one's own mathematical understanding to the student. However, even in the worst case the aim is to uncover student thinking. Thus, asking for explanations and posing questions that reveal nuances and subtleties of the mathematics is always a ready technique in an ETI.

---

[3] Thanks go to Kevin Moore for suggesting this name.

An ETI is always approached with the anticipation that the student may not interpret the text of the problem in the same way as the interviewer, and so the interviewer's first task is to ask questions about the problem statement, in order to create a model of the mathematical problem that the student is working with. As the interview develops, the interviewer creates hypotheses[4] about the student's understanding of the problem and of the mathematics, and the relationship between those hypothesized understandings and the difficulties that the student appears to have. Based on these hypothesized understandings, the interviewer suggests to the student, at an appropriate moment, new ways of interpreting the problem. "Appropriate" typically means that the interviewer has judged that the student is at a dead end or is on a reversibly unproductive path of reasoning. The interviewer's role is to create contemporaneous hypotheses about the student's particular way of thinking or particular understanding of the problem that is causing the difficulty, and to form a suggestion or alternative task based on that hypothesis. These suggestions comprise the "teaching" component of the interview, although the choice of the term "teaching" is slightly misleading here, in that the interviewer's goal is to create and test a model of the student's mathematics, not to bring the student to a particular understanding or to insure that the problem is completed. Rather, the purpose of these suggestions is to generate more data about the student's mathematics. If the suggestion enables the student to easily complete the problem, or the nature of the student's difficulties changes, then the interviewer has information about the new way that the student understands the problem now, and how the student understood

[4] Steffe and Thompson (2000) also describe the generation of on-the-spot hypotheses while engaged in the teaching component of a teaching experiment.

the problem previously. If the nature of the student's difficulties remains the same, then the interviewer needs to revise his or her model.

The design of the teaching episodes in this teaching experiment in exponential functions was based on a formalization of the ETI, adding the components of Steffe and Thompson's methodology that had been omitted from the informal exploratory teaching interviews: in particular, a stronger emphasis on making the teacher-researcher's hypotheses explicit, and the inclusion of a witness.

Furthermore, in this teaching experiment, where the teaching episodes have a serial nature, the word "teaching" itself has a different meaning than in an ETI. Due to the serial nature of the teaching episodes, I also committed to bringing the students to particular understandings of the problems that I ask the students to engage in, as later problems built on those understandings. So the "teaching" in teaching experiment took on two different meanings. One meaning is the suggestions that students think about the problem in particular ways in order to investigate hypotheses about the students' understandings, and the second meaning is to suggest that students think about the problem in particular ways in order to develop the students' understandings toward a learning goal. The methodology for the second meaning is described in a later section, entitled "Hypothetical Learning Trajectory."

The Teaching Agent

The role of the teaching agent, or teacher-researcher, was played by myself – denoted "Carlos" in later transcript excerpts. Although I had played the teacher-researcher in a number of ETIs, this was the first time I played the role in a teaching experiment that had multiple teaching episodes building upon one another. In this situation, the most difficult aspect for me was learning to balance the role of "teacher" and the role of "researcher," something that I did not accomplish entirely successfully.

The teaching experiment which I have described so far has at its heart two very different goals: A research goal, which is to create models of the students' individual mathematical understandings, and a fundamentally different instructional goal – that the students learn the Verhulst model from a covariational perspective. These two types of goals are not fundamentally incompatible, but in situations where time is limited, these goals select for the teacher-researcher different choices of action: is time to be spent on asking the students to explain in more detail their own meanings, or is time to be spent on changing the students' understanding of the problem? In this sense, one goal must be prioritized over the other.

As this study is about the interactions between covariational reasoning, exponential reasoning, and differential equations reasoning, the highest priority was always placed on modeling the students' mathematics. I anticipated that analyzing the interactions between a student's mathematical ideas would be impossible without a robust model of the mathematical meanings and ways of thinking that were at the root of students' behavior. However, I was also aware that studying the students' difficulties with the Verhulst model would also impossible without a good faith effort to teach it.

The outlook that I took in designing this experiment then, was to plan for teaching the Verhulst model, with the awareness that for a combination of research, pedagogical, and curricular commitments, I must first focus on the students' learning of exponential growth. This focus affected both the teaching experiment design and methodology.

From a design point of view, the teaching experiment became a teaching experiment in exponential growth, teaching toward the Verhulst model, but with the anticipation that the exponential growth portion of the teaching experiment would take the bulk of the time, to the point that the Verhulst model might not be covered at all. The details of this design are in a later section, "Hypothetical Learning Trajectory." Fortunately, there was enough time available to study the Verhulst model with one of the two students.

Methodologically, I chose to place a greater priority on modeling the student's mathematics in the moment rather than on insuring that the students reached a particular understanding. By choosing to err on the side of modeling rather than on the side of instruction, my intervention differed from the exploratory teaching interview methodology described above. My questioning was more often in the form of asking for clarification, or suggesting situations to which students could apply their reasoning so that both the students and I could study the consequences of their reasoning, rather than suggesting particular ways of thinking about situations.

In retrospect, I believe that I erred too far on the side of caution in this teaching experiment, choosing to question further in situations where more declarative suggestions would have been more helpful to the students. A lack of confidence on my part due to this being my first teaching experiment was certainly a major factor in these choices.

The Class

The role of the teaching agent is to generate hypotheses about students' understandings and act to test those hypotheses in real time. Forming these hypotheses, particularly about the students' meanings of covariation requires attending to the students' words and actions at a fine level of detail. I chose two students because I believed that a larger number of students would strain my abilities to form hypotheses and attend to student thinking at the level of detail necessary for this experiment. To know Tiffany and Derek one must also know about the course in which they participated and have an idea of the nature of their participation and the mathematical understandings they developed.

As mentioned previously, the two students in this study were selected from a high school Algebra II class that was led by the regular teacher, but designed in collaboration with researchers from Arizona State University. This Algebra II course covered all the regular material (students took the same district final exams as all other Algebra II students), but it was designed with the intent of teaching students Algebra II from the perspective of continuous covariation – including a deliberate efforts to increase the amount of time the students spent reasoning with graphs.

Prior to the teaching experiment in March, Derek and Tiffany had covered the majority of the material in the course. The focus of the first semester was primarily linear functions. The course began with an introduction in unit conversation, which later instruction used to teach a sense of very large and very small numbers, fractions, and rate of change. In particular, the students were asked to convert various speeds (miles per hour) and rates (miles per gallon) to other units (e.g. feet per second, liters per kilometer).

From rate conversations, the students transitioned to modeling linear behavior in graph and equation forms. Students employed Pacific Tech's *Graphing Calculator* software to use parametric functions to create animations of points moving along paths in the coordinate plane. Students continued to use the *Graphing Calculator* in visualizing systems of linear equations of two variables, and linear inequalities in the coordinate plane.

The second topic of the first semester was quadratics, taught from the perspective of rate of change. Students were presented with a graph of a constant function, and asked to graph the original function for which the constant function would represent the rate of change (Figure 5), essentially asking the students to perform a casual form of integration. In Figure 5, students were given a graph of a function's rate of change and were asked to sketch a graph of a function that had that rate of change. The rate of change graph (ROC) in Figure 5 shows a function's rate of change as having -3 as a constant value. The Function graph in Figure 5, created by Tiffany, shows her attempt to capture this by constructing a linear function having -3 as a rate of change and passing through the point (0,3).

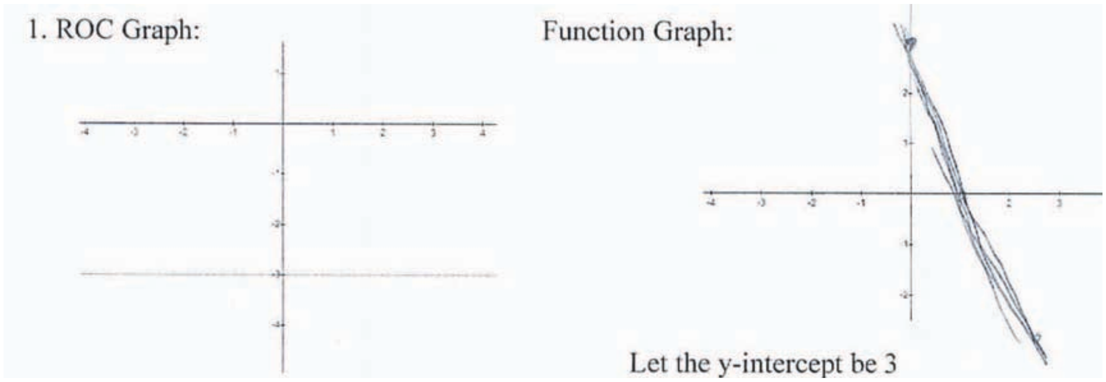


Figure 5. “Tiffany's” solution to an informal integration problem.

This process was extended (briefly) to a step function, and then to integrating a line (Figure 6). The left graph in Figure 6 shows a rate of change function that is a step function whose “steps” increase in the *y* direction at a rate of 2 times the increase in the *x* direction. The function having this step function as its rate of change function has a constant rate of change of -4 between -2 and -1.5, -3 between -1.5 and -1, and so on. The graph on the right is Derek’s sketch of a function that has this rate of change and which passes through the point (0,0). (Derek’s graph does not reflect a rate of change of 0 between 0 and 0.5.)

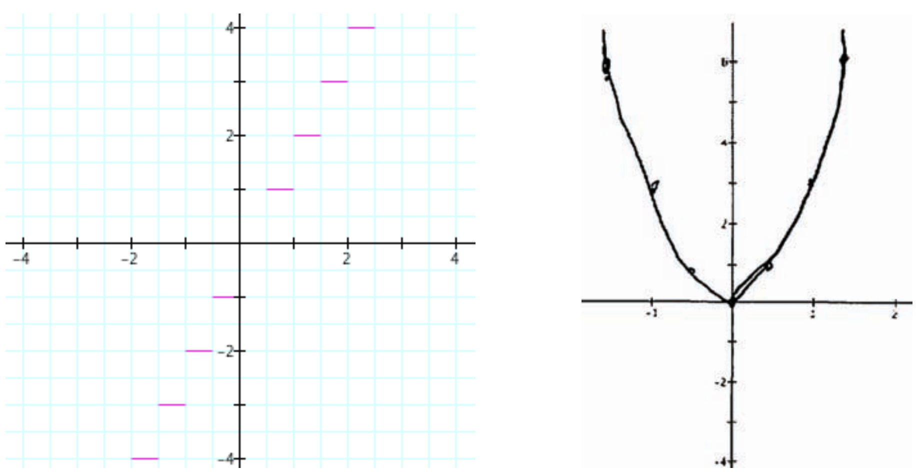


Figure 6. "Derek's" sketch of the “original function” given a step-wise linear "rate of change function."

The students then began studying quadratics symbolically, learning to manipulate the standard form of a quadratic by completing the square and finding the vertex form of a quadratic. The semester concluded with a unit on sequences and series.

The second semester opened with lessons in building exponentiation as repeated multiplication and having the students derive laws of exponents from repeated multiplication meanings of exponentiation. The students then applied these laws in simplifying expressions containing exponents and radicals. The student's study of polynomials began with presenting the students with graphs of functions on numberless axes, and asking the students to graph the sums of those functions, forcing the students to judge the value of each function by using the height of the function as the student imagined moving that height from left to right. From here, the students learned to graph polynomial functions as the dilations and sums of the graphs of monomial functions.

Function composition was taught in a similar manner, with the students first learning to graph the composition of two functions presented as graphs, and then moving to symbolic composition. Function inverse was taught as an extension of function composition. The final lessons in polynomials prior to the teaching experiment asked the students to use the distributive property to convert polynomials from factored form to standard form. The students also learned to use factored form to graph polynomials based on their roots.

At the time of the teaching experiment, Tiffany and Derek had already engaged in the class as described above. I observed their classroom participation, in person, every day. Their classroom discussions were recorded on video, their student work was recorded and scanned, and they had participated in exploratory teaching interviews. I selected Derek and Tiffany on the basis of their recorded participation in class, asking them to participate in a series of interviews involving exponential growth. Both students volunteered immediately.

The Students

Derek and Tiffany were two of the three highest performing students in the class, and I selected them both because I knew the teaching experiment would be mathematically challenging, and because I had observed during class that Derek and Tiffany had very different approaches to thinking about covariation. Tiffany preferred to vary quantities in equal sized steps, while Derek imagined quantities changing continuously, a situation that I modeled as analogous to the differences between Confrey and Smith's (1994, 1995) meaning of covariation and Saldanha and Thompson's (Saldanha & Thompson, 1998; Thompson, 2008b) meaning of covariation.

As an example of their difference in behavior, Tiffany would always begin by plotting points equal distances apart (usually plotting on the whole numbers, or increments of 0.5), while Derek would plot points at irregular intervals, placing more points in areas where he wanted better resolution into the behavior of the function. This is most clearly seen in the graphical sums of functions unit, and I've included sample work here (Figure 7 and Figure 8).

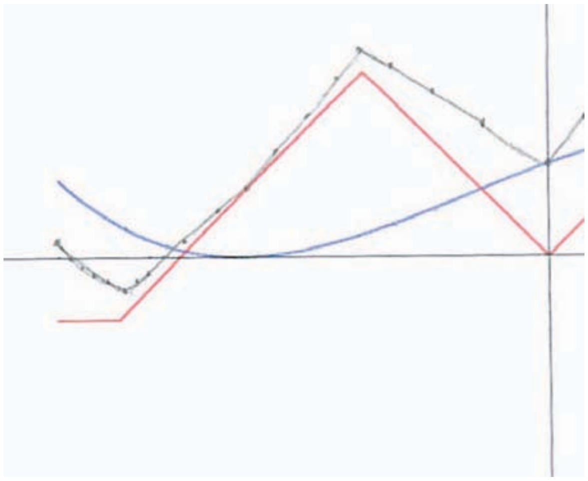


Figure 7. A portion of Derek's graph of the sum of the blue and red functions.

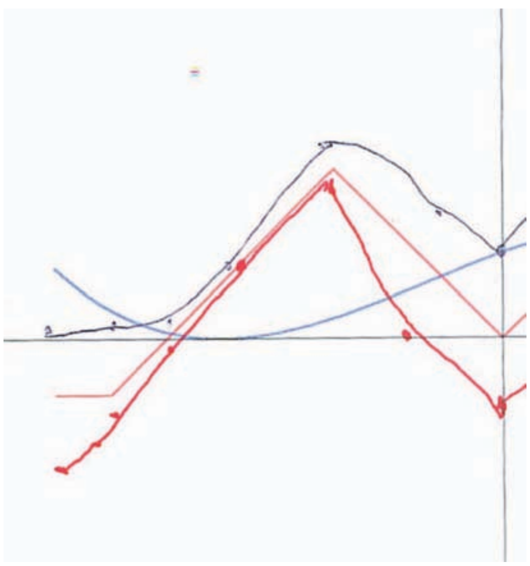


Figure 8. Tiffany's solution to the sums of function problem (black dotted line), and Tiffany's solution (red dotted line) to a "challenge problem" of finding the difference between the red and blue functions.

Note the higher density of points to the very left, near the first bend, in Derek's graph. This is in contrast with Tiffany's solution to the same problem (Figure 8). Tiffany graphed the sum of the red and blue functions in black. Note that Tiffany's points are more evenly and more widely spaced, and as a result, the some of the details of the solution are missing. Tiffany's red graph (also Figure 8) is a graph of the difference of the two functions (red minus blue), and is an example of a challenge problem given to students who finished early.

Figure 8 is typical of Tiffany's approach to covariation. She frequently used widely spaced points in graphing. When numbered axes were available, these points were nearly always spaced at 1 or 0.5 increments (Figure 9).

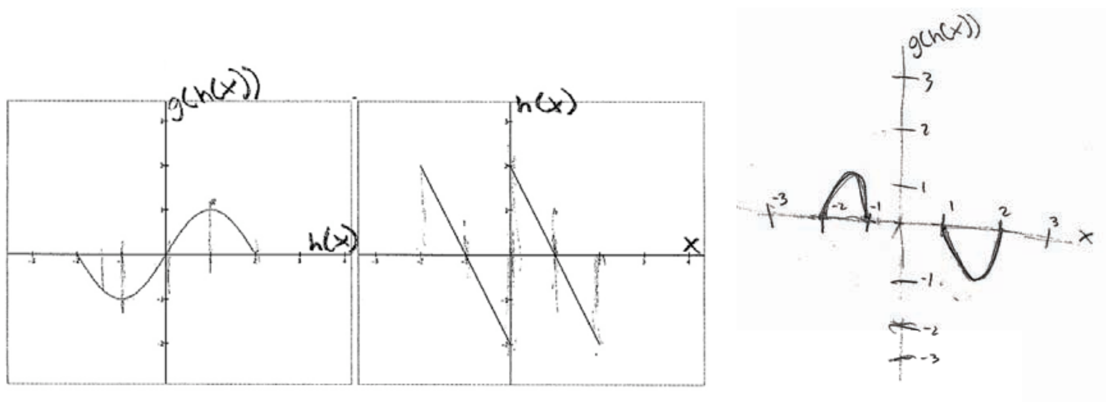


Figure 9. Tiffany's solution to graphing the composition of functions h and g.

It is important to note in Figure 9 the lines Tiffany drew on the upper right graph. This reflects Tiffany's interpretation of the graph in whole valued increments of x. As a result of her incremental approach, Tiffany correctly finds the values of g(h(x)) at x=-2,-1,0,1, and 2, but does not fill in the behavior of the composition graph between -1 and 1.

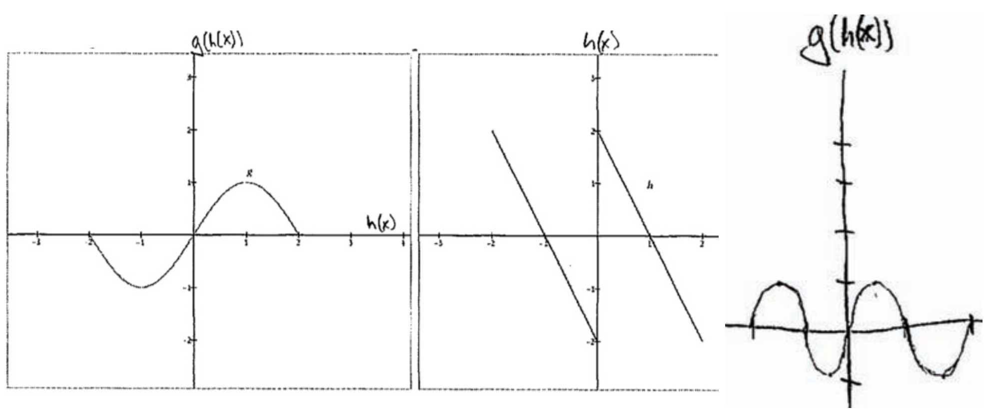


Figure 10. Derek's solution to the composition problem (cropped for space).

Contrast Derek’s solution to the same problem (Figure 10) with Tiffany's solution above. Derek shows no scratch work at all, lacking even points to mark out the graph. Derek's solution consists only of the graph and required axes labels. This is typical of Derek’s approach to problems in the course. Derek worked extremely rapidly and intuitively, frequently showing no work at all.

As two of the highest performing students in the class, Derek and Tiffany frequently finished their work early. As a way of keeping these students occupied, I often gave them “challenge problems.” These were problems made up on the spot that took the ideas involved in the work they just did and pushed them slightly farther. An example of one such challenge problem is shown in Figure 8, in which I asked Tiffany to extend her sums of functions reasoning to find the difference of two functions. Her solution can be seen as the red dotted line. Another example of a challenge problem is Figure 11, in which Derek was asked to graph the sum of two functions with discontinuities.

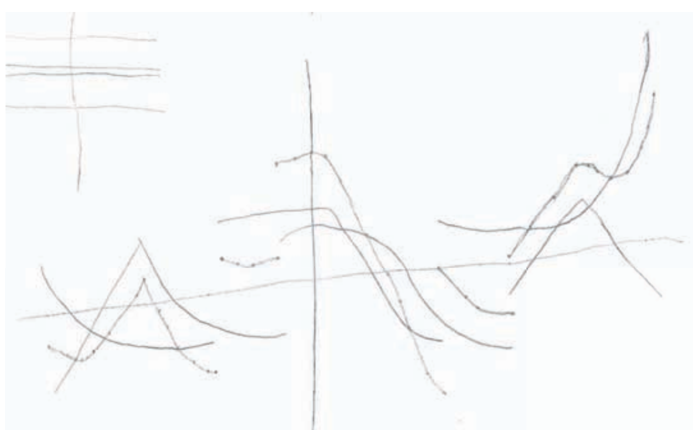


Figure 11. Derek's (dotted line) solution to the "challenge problem," in graphing the sum of two given functions (solid lines).

I designed Derek’s challenge problem (Figure 11) as a problem that was similar to the sums of function problem he had just completed, (Figure 7). The challenge aspect comes from the increased complexity of the two functions I assigned him to sum – including multiple jump discontinuities and rapid changes in qualitative behavior of the functions. Derek had very little difficulty with this problem, indicating a robust methodology. We can see in Figure 11 that Derek used the same method of spacing points unevenly. He placed points with greater resolution in areas where he felt the need for more detailed information about the behavior of the function he was asked to find.

Derek – who finished early more often than any other student – showed great enthusiasm for these challenge problems, and when pressed for class time, would take them home and complete them overnight, even though he received no extra credit for them. Other students in the class, including Tiffany were also intrigued by the challenge problems, and would give them a try themselves or ask for challenge problems of their own.

Overall, I selected Derek and Tiffany for a broad variety of reasons: I knew that they would be willing to participate; I had confidence that they had the personal disposition and mathematical aptitude to deal with the challenging mathematics that they would encounter. These characteristics were particularly noticeable in their attitudes towards the challenge problems. Secondly, my experience with them in the classroom led me to anticipate two very different systems of thinking about change that paralleled my interest in the different covariational frameworks of Saldanha and Thompson and Confrey and Smith.

The Witness

Steffe and Thompson (2000) describe the difficulty of the role of the teaching-researcher (teaching agent) as being one where the teacher-researcher must step out of the interaction, reflect on it, and take action on that basis. Because the teacher-researcher is engaging in the interaction in real time, this requires the teacher-researcher to be thinking about the interaction from two perspectives at once. The role of the witness is two-fold: to aid in creating an outside perspective by always being outside the interaction, which enables the witness to observe elements of the students' actions that the teacher-research might have missed, and to challenge the teacher-researcher's models of the students on the basis of that committed outside perspective.

To this end, many of the teaching episodes of this teaching experiment were witnessed, and those teaching episodes were followed by a debriefing session in which the witness, Pat, and myself discussed both my own and the students' actions during the teaching episode, shared our interpretations of those actions, and collaborated on the planning of the subsequent teaching episode.

The Record

Every teaching episode was recorded on videotape, and all of the students' and my own written work were collected and scanned at the end of every teaching episode. Tiffany, Derek, and I all wrote in different colors, so that communal written work could be attributed to each writer. Furthermore, in cases where the students were continuing from and writing on previous work, the students wrote in a different pair of colors, so that later work was distinguished from previous work. In the teaching experiment, Carlos always wrote in green, Derek always wrote in black or purple, and Tiffany always wrote in blue or orange.

From these recordings, I created transcripts of every teaching episode, in multiple passes. The first pass of transcription frequently occurred on the day of the teaching episode, as part of planning the subsequent teaching episode. Doing immediate transcription gave me the opportunity to familiarize myself intimately with what was said by whom during each teaching episode, and allowed me to revise my hypotheses on that basis prior to the subsequent teaching episode.

The students also used the Pacific Tech graphing software *Graphing Calculator* for arithmetic calculations and to generate graphs of functions that they created during the teaching experiment, and I occasionally used the same program for animated demonstrations. Each Graphing Calculator file used by either myself or the students was saved, with calculations and function definitions intact.

The role of notation. Part of the recording design was that the students were given a certain amount of freedom to choose their own symbol systems. The students' development of a notational and a symbolizing system for recording their situational understandings was something that I both anticipated and encouraged as a critical part of the modeling process for the students (Gravemeijer, et al., 2000), and also a critical part of the process of my own modeling of the student's mathematics. However, as part of my instructional agenda, I did constrain the students to particular broad categories of representation, asking the students to "write a function" or "draw a graph" rather than leaving the choice of graphical or symbolic representation entirely up to them. This constraint made it easier for me to anticipate possible responses and to plan a sequence of activities.

I anticipated that the student's choices of symbolic notation in particular would give me insight into the way that the students were imagining the quantities and the interactions between quantities. As an example: if as student chose to represent rate of change in function notation r(p), this suggests to me that the student might be imagining rate changing, and that rate is calculated as a function of population. This is as opposed to, for example, the notation r(t), which also suggests to me that the student might be thinking of rate as changing, but that the student is think of rate and population changing parametrically in time rather than imagining rate as calculated from population.

At the same time, I took care not to read too much into the students' notational choices. If a student chose to represent rate of change as simple r, this would not tell me whether the student was thinking of rate as variable or constant, or whether the student thinks that rate depended on other quantities or not. In these situations – when I felt need for clarification necessary – I asked clarifying questions to establish what the student meant by r, the units that the student imagined r being measured in, and what relationships among the quantities the student was imagining, so that the student's meaning of r was recorded in video along with the student's notational choice of r on the scanned page.

The software the students used has its own notational conventions as well, and these conventions reduced the freedom that the students had to come up with their own notation. In situations where I anticipated that the students would need to use Graphing Calculator in the future, I introduced the software's notational conventions explicitly.

Conceptual Analysis

The design of the instruction,[5] the on-the-spot hypotheses, and the retrospective analysis were all based on von Glasersfeld's (1995) notion of "conceptual analysis." Von Glasserfeld's definition of "conceptual analysis" is rooted in his meaning of the linguistics term "conceptual semantics." In describing conceptual semantics, von Glasersfeld says:

> [Conceptual semantics] does not try to find appropriate verbal definitions of words, as one might find in a dictionary, but instead, aims at providing 'recipes' that specify the mental operations that are required to obtain a particular concept (Glasersfeld, 1995, p. 76)

From this passage, I take the meaning of conceptual analysis of a mathematical idea to mean "providing a 'recipe' that specifies the mental operations that are required to reason in a particular way." As an example, my conceptual analysis of a Thompson constant rate at this point in the design included reasoning about two magnitudes that are imagined to be accumulating simultaneously over arbitrarily small intervals, in such a way that any accrual in one magnitude is always proportional to the accrual in the other magnitude. In this way, a Thompson rate depends on particular images of varying magnitudes, particular images of covariation, and a particular notion of proportion.

---

[5] See the later section Hypothetical Learning Trajectory

Related to the notion of the conceptual analysis of a concept is the notion of the conceptual analysis of an individual. Rather than identifying the meaning of a particular concept, conceptual analysis of an individual involves identifying the meanings that an individual holds: what their concepts are. Von Glasersfeld describes conceptual analysis of an individual as asking the question "what mental operations must be carried out to see the presented situation in the particular way one is seeing it." (Glasersfeld, 1995, p. 78) Or, in order to make the researcher's position in this question clear, I would be asking the question "what mental operations might my subject have carried out in order to see the presented situation in the particular way that he appears to me to be seeing it?" It is these hypothesized mental operations that will comprise my models of the students' mathematics.

Conceptual analysis of a concept and conceptual analysis of an individual are related. Although I have suggested that a concept has an independent existence, in fact a concept resides in the mind of an individual. In the case of the design of the instruction, the individual was an epistemic student, a hypothetical student that I anticipated learning this material. In the case of the retrospective analysis, the individuals were the students Derek and Tiffany themselves.

Retrospective Analysis

Steffe and Thompson emphasize retrospective analysis of the records as a "critical part of the methodology" (Steffe & Thompson, 2000, p. 292). For this study, I used a multi-pass system of retrospective analysis. The first pass was more accurately a hybrid of on-the-spot and retrospective analysis, specifically the creation and examination of the transcripts of the videotapes of each teaching episode prior to the subsequent teaching episode. This pass occurred daily, as soon after each teaching episode as possible, and consisted of digitizing and creating a first pass transcription of that day's teaching episode video. The process of transcribing forces one to attend to minute details of interactions in order to develop a coherent and accurate record. I saw the transcription of the videotapes as being a critical method of familiarizing myself with the students, and critical to basing my planning and hypotheses for the next teaching episode.

At the end of the teaching experiment, I compared the first-pass transcriptions to the video again, and further revised for accuracy and detail. The second pass of retrospective analysis occurred after all the teaching episodes had been transcribed. At that point, my perspective of the teaching episodes was based in having the entire story, rather than the story as it developed. This second pass consisted of converting the first-pass transcripts to subtitles to the videos, and viewing each video with those subtitles as a method of both reviewing what had occurred and catching transcription errors. In this second pass, I began to identify "themes" in the stories of the students, and to use those themes to focus the study of the transcripts themselves.

With this added perspective of having an image of what happened later, I re-examined the transcripts of each episode, using the "themes" I identified in my second pass to focus my attention on the line-by-line details of the original transcripts. Here the emphasis was on studying how the themes I identified interacted in a very specific way: by creating models of the students' covariational reasoning, exponential reasoning, and understanding of the Verhulst model over time.

Hypothetical Learning Trajectory

A teaching experiment, as described by Steffe and Thompson, consists of the teaching agent playing two roles simultaneously: teacher and researcher. In the above description of the teaching experiment, I focused on describing my primary concern: how I will form models of the student's mathematics. This second focuses on the teacher role of the teacher-researcher, discussing the instruction that I originally planned. Simon and Tzur (2004) define a hypothetical learning trajectory as consisting of the goal for the students' learning, the mathematical tasks that will be used to promote the students' learning, and hypotheses about the students' learning process. In a teaching experiment, the hypothetical learning trajectory can also act as a record of the teacher-researcher's thinking prior to the teaching experiment, so that the teacher researcher can see how his or her own thinking was changed by the process of studying student learning.

In my own the case, I followed the sequence of tasks which I will outline fairly faithfully, but the hypotheses and goals that began the teaching experiment were not the models and goals I had at the experiment's end.

Hypotheses and goals

At the time that I designed the teaching experiment, the central hypothesis of my instructional design was that a sophisticated understanding of the Verhulst model is obtainable with a very small but carefully chosen toolkit of meanings and ways of thinking. Synthesis of covariation and the Verhulst model described in the previous chapter does not require any understanding of the Calculus. There is no integration or differentiation, and only a very rudimentary understanding of limit. Likewise, both the

Verhulst model described there and exponential reasoning described by Thompson (2008a) are treated without any exponentiation, laws of exponents, rational exponents, or logarithms. The conceptual semantics of the Verhulst model requires only a few tools: an understanding of the modeling contexts, the mathematics of quantity (Thompson, 1988, 1990, 2008b) the mathematics of continuous variation (Saldanha & Thompson, 1998; Thompson, 2008b), and some facility with graphing and manipulating linear functions. The only exceptions to this are the graph of the parabola on the phase plane, a recognition that functions other than linear functions could potentially exist, and the concepts of covariation, constant rate of change and per-capita rate of change, and the idea of phase plain analysis that I intended to build along the way.

My learning goal for the students was that they develop a meaning of the Verhulst model – where meaning here is used in a conceptual semantics sense, that is that the students engage in the mental operations that comprise an understanding of the Verhulst model. I anticipated that the concepts that students were to construct during this instruction -- covariation, rate, per-capita rate, and phase plane analysis – would be ways of reasoning that students could additionally apply to a wide variety of problems. So this learning goal comprised a number of learning and instructional sub-goals:

1. That the students come to imagine covariation as the coordination of two quantities that each vary in a dense number system.
2. That the students come to imagine rate as a proportional relationship between two quantities – and associate that image with linear growth.

3. That students come to imagine per-capita rate as the constant rate accumulation an individual's direct children.
4. That the students imagine as a result that rate of change of the entire population of identical individuals is non-linear.
5. That students come to able to construct phase plane graphs of the value of a function and its rate of change.
6. That the students learn to how to interpret phase plane graphs in such a way that they can read off the behavior of the original function.

Tasks

The sequence of tasks that I chose to use directly follows Thompson's construction of the exponential (Thompson, 2008a). Thompson begins with simple (linear) interest, then progresses to imagining piecewise linear functions as a result of compounding, and finally functions that result from allowing the regular compounding interval size to approach zero.

The sequence of tasks that I designed for this teaching experiment begins precisely as Thompson does, with simple interest followed by compound interest. However, I chose to deal with exponential growth (the limiting case) differently. Anticipating that I would be using a per-capita rate and phase plane approach to teaching the Verhulst model (Bergon, et al., 1996), I chose to introduce the ideas of per-capita rate and the phase plane at the same time as I introduced the exponential function, and then drawn connections between exponential growth and compound interest growth in the phase plane, rather than as the limit of the piecewise linear compound interest function.

The second disparity between Thompson's approach to exponential growth and Bergo, Harper and Townsend's approach to deriving the Verhulst model is that Thompson's approach occurs in a financial context (Thompson, 2008a) while the Verhulst model is typically taught in a biological context (Bergon, et al., 1996). The biological model has the disadvantage the number of people is inherently discrete, while financial models are more easily thought of continuously, but a financial motivation for the Verhulst model is unrealistic. In order to bridge this disparity, I chose to begin with a financial context, and then design a transition task to the biological context, asking the students to evaluate the suitability of the financial model in a biological context. In making this transition, the distinction that Gravemeijer (2000) makes between a *model-of* a situation, and a context free *model-for* mathematical reasoning was one that particularly influenced the design of the tasks. In developing equations and graphs to represent their understanding of various account policies, the students would be creating *models-of* financial growth, and in making the transition to a biological context, the students would have to reconceptualize their *model-of* financial growth to a *model-for* reasoning about various contexts, including biological contexts. Gravemeijer refers to this as a model taking on a life of its own – the behavior being modeled existing independently of the context.

Thus the sequence of tasks ultimately became: simple interest, compound interest, per-capita interest, limiting compound interest in the phase plane, phase plane analysis of exponential growth, transitioning to the Malthus model for exponential growth in biology, deriving the Verhulst model, and phase plane analysis of the Verhulst model.

Prior to the start of the teaching experiment, I constructed an outline for each task in the sequence, consisting of the wording of the task; a list of instructional goals describing the purpose of the task; a list of hypotheses or anticipated difficulties describing how I anticipated the task would play out with my students; and a sequence of questions to ask the students, based on the hypotheses and goals listed above. The full outline for each task is included as an appendix, and summaries of the outlines are included below.

The design of these tasks and the outlines, were in part a collaborative effort with Patrick Thompson, who would also serve as the witness of the teaching experiment. "Pat" served a similar role as a "witness" in the design of the teaching experiment: playing the role of an outside observer who did not necessarily hold the same commitments of meaning that I did. In this way, Pat was able to identify ambiguities in wording in both the tasks and the outlines, as well as anticipate difficulties the students would have that I overlooked as a result of taking the meanings in my own head for granted. This revision process was a critical part of the design of the tasks and the hypothetical learning trajectory, in that it insured that the design was focused towards having a dialogue with someone other than myself, anticipating the discussion I would be having with the students.

Although the outlines included lists of questions, in the actual teaching experiment I used my prepared questions very rarely, preferring to build directly on the statements the students themselves made. Steffe and Thompson describe the role of the teaching agent as one where the agent would have to make on the spot hypotheses and react based on those hypotheses. As a result, the questions that I asked in the field mutated rapidly from the sequences listed in Appendix A. As discussed previously, my primary concern was to build models of the students' mathematics, and the questions that I asked in practice were primarily questions of clarification, or asking students to carry their reasoning further to examine a consequence of that reasoning.

The Simple Interest Task

> Jodan Bank uses a "simple interest" policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of 8% of the initial investment per year. Create a function that gives the value of an EZ8 account at any time.

The simple interest task was intended to be an introductory task, placing the mathematics the students were already familiar with (linear modeling), in the context that would be used for the rest of the experiment, and also to provide me with an opportunity to discuss a dynamic situation with students as a way of forming an early model of the students' ways of thinking about variation, rate, and time. I also wished to establish early on a distinction between the current value of the investment and the initial value of the investment, as preparation for compound interest in which the students would have to make a distinction between the value of the investment and the value of the interest earning portion of the investment.

My goals for this task were introduce the (bi)linear model of financial growth from which Thompson (2008a) constructs the exponential from, to introduce the idea of a parameter by having students work with multiple starting investments, to model the student's reasoning about time, and to establish a meaning of "growth rate" as being the number of dollars per year corresponding to the "slope" of the linear function model.

Compound Interest

> The competing Yoi Trust has introduced a modification to Jodan's EZ8, which they call the YR8 account. Like the EZ8 account, the YR8 earns 8% of the initial investment per year. However, four times a year, Yoi Trust recalculates the "initial investment" of the YR8 account to include all the interest that the customer has earned up to that point.
>
> Fred deposited $1250 in a YR8 account. Create a function that gives the value of Fred's YR8 account after any amount of time (measured in years) since he made his deposit (assuming he makes no deposits or withdrawals).

The purpose of this task was to begin a transition from linear function models to exponential function models by introducing compounding. In the compounding described above, the function is at all times growing linearly; however, the rate of growth of the linear function changes at fixed intervals (the instant that Yoi trust recalculates the "initial investment"). If we make a distinction between the number of dollars earning interest (principal), and the amount of money that is reported to Fred (the account value), then the account value grows linearly except at the points every quarter year when the principal changes, and the principal grows geometrically.

My goal for the students with this task was to have the students create the function for the account value in intervals, imagining that each compounding period is a new investment in which they can use their simple interest model, and then by finding the value of the investment at the end of the compounding period, be able to use that new principal in the linear model for the next compounding interval.

I also intended that the students be able to “see” the “classic” compound interest formula $A(t) = P(1 + \frac{r}{n})^{nt}$ – the geometric model for principal growth – by examining the pattern of their calculated principals, when those principals are left in unevaluated (open) form.

Per-capita Exponential Model

> The Savings Company (SayCo) also competes with Yoi Trust and Jodan. SayCo’s PD8 account policy is as follows: if you have one dollar in your bank account, you earn interest at a rate of 8 cents per year. For each additional dollar, your interest increases by another 8 cents per year. If you have fractions of a dollar in your account, your interest increases by the same fraction, so 50 cents earns interest at 4 cents per year. Here is SayCo’s new feature: At any moment you earn interest, SayCo adds it to your account balance; every time your account balance changes, SayCo pays interest on the new balance and calculates a new growth rate. Why is SayCo’s PD8 the most popular account?

The design of this question was based on a difference in meaning between Pat and myself. Specifically, the original form of this task did not include the statement “Here is SayCo’s new feature” or the sentence that follows. I saw, and still see “SayCo’s new feature” as being inherent in the original policy statement, which is intended to be a verbal representation of the formula $\frac{dy}{dx} = .08x$.

The image that my conception of the original description depends on requires imagining three continuously changing quantities: The number of dollars in the account, the number of years since the investment began, and the rate of change of the first quantity with respect to the second. In this image, the intended meaning of the above passage is that for any point value of the time quantity, there is an associated value of the rate quantity, based on the value of the dollar quantity at that time, and the value of the rate quantity is proportional to the value of the dollar quantity with a constant ratio of .08. So, because the number of dollars is changing continuously, and the rate of growth depends only on the number of dollars, the rate is also changing continuously.

The addition of Pat's "SayCo's new feature" depends on a different interpretation of the original statement. This second perspective is based on the interpretation of the original statement through the lens of a compounding model. Imagine two continuously changing quantities: the number of dollars in the account, and the number of years since the investment began. If one imagines starting with a certain number of dollars at year zero, there is a certain associated growth rate for that number of dollars, and then at some point in the future, the account is compounded. At this compounding point, there is a value of the time quantity, a corresponding value of the dollar quantity, and a new rate is calculated. Contrast this interpretation with the first interpretation, in which there is no imagined compounding action at all, but is simply an image of a proportional relationship between two quantities changing continuously in parametric time: whatever the amount in the account happens to be at any moment, the rate at that moment happens to be .08 times that, no compounding involved.

In the discussions around this task, I was adamant that the correct interpretation of the original task could only by the first (my) interpretation, and Pat was equally adamant that because he had interpreted the task in the second way, the wording of the task must therefore be ambiguous. Since Pat established the ambiguity with an irrefutable proof of the existence of a second interpretation, the addition of “SayCo’s new feature” was included to clarify that the account was being compounded continuously.

Anecdotally, some time after the teaching experiment, I presented the original form of this task to a group of six mathematics education graduate students. Every single one of those students agreed that the problem was insoluble because it did not state how frequently the account was compounded, indicating that every single one of the six interpreted the task statement (without the “SayCo’s new feature” portion) in the second way. To this day, I have yet to determine a phrasing of this task that evokes the first situation in anyone other than myself, and the examination of these differences in interpretation and the reasons for them is something I wish to explore in the future.

On the basis of Pat’s argument for ambiguity, and the fact that the students were working this problem immediately after a compound interest problem, We designed the task around the “SayCo’s new feature” phrasing, although I retained some misgivings that the “SayCo’s new feature” phrasing of the task would make a discussion of continuous compounding more difficult, rather than less difficult, by describing a process of recalculation. The phrasing “every time the account value changes” evoked in me an image of a previous and a later value, which would imply discrete compounding. I anticipated that the students would have a similar difficulty: that students would imagine that after some tiny bit of time passes, the balance updates with earned interest and the growth rate updates accordingly; in contrast to: whenever the amount invested changes, even by fractions of a penny, the balance updates with earned interest and the growth rate updates. Much of the planning of this task was around the goal of helping the students imagine continuous compounding.

A second idea that this problem was intended to introduce is a particular image of the relationship between constant per-capita growth rate and proportional population growth rate: That every single dollar contributes at a constant rate of .08 dollars per year, and that the population of dollars as a whole contributes at an aggregate rate of .08*population dollars per year. This relationship is based on the distributive property: as the dollars are added together to form the population, the per-capita growth rates are also added to together to form the population growth rate, and a factoring out of the .08 from the sum of per-capita growth rates results in .08 times the sum of individuals, or .08*population. One representation of the full chain of reasoning, including units, is shown in the equation below.

$$\begin{aligned}&\frac{.\$08}{yr}+\frac{\$.08}{yr}+\frac{\$.08}{yr}=\frac{\frac{\$.08}{yr}}{\$}(\$1)+\frac{\frac{\$.08}{yr}}{\$}(\$1)+\frac{\frac{\$.08}{yr}}{\$}(\$1)\\&=\frac{\frac{\$.08}{yr}}{\$}(\$1+\$1+\$1)=\frac{\frac{\$.08}{yr}}{\$}(\$3)=\frac{\$.24}{yr}\end{aligned} \tag{16}$$

The ultimate goal for this task was to have students create a variable for the rate of change of the account, and represent the relationship between the rate and the value of the account with an equation in their own notation, essentially creating the differential equation form of exponential growth.

Limiting Compound Interest

Previously: Yoi Trust one-upped Jodan's EZ8 account, which was a simple interest account, with their YR8 account, by adding earned interest to the account value at the end of every 3 months (4 times per year).

For the Yoi YR8 account, create a function that gives the rate of change of the value of the account (in dollars per year) for every value of the account.

The ultimate goal of this task was to introduce two ideas: the phase plane, and that the SayCo policy above resulted from a limiting process of increasing the number of compounding actions per year. The design of this task was based on the fact that earlier in the year, the students had worked with functions defined parametrically.

I envisioned asking the students to construct the phase plane graph of the YR8 account point by point, using the equation they created in the compound interest task. At each moment in time, the student would be able to use that equation to calculate the number of dollars in the account, and also "read off" the rate of change as the *m* in the $y=mx+b$ form of the corresponding "piece" of the piecewise-linear compound interest function. Given those two data, the students would then be able to plot a point in the phase plane, and repeat this process until there were enough points to establish that the compound interest function was a step function in the phase plane: maintaining a constant rate of change until a compounding event occurred, and then "jumping" to the next constant rate of change.

I anticipated that a key observation for the students to make would be that each compounding action would be represented on their graph by a discontinuity, and in that way, the students would be able to use the graph to tell time by counting the number of discontinuities as the number of quarters.

This observation would be necessary to achieve the second goal that I had for the task, which is for the students to imagine how the graph of compound interest in the phase plane would change as the compounding interval changed, becoming – in the limit – the graph of a line, with the equation y=.08x, the same equation that the students found to be the relationship between rate and amount in the previous task.

Phase Plane Analysis

> Create a graph showing the relationship between the growth rate of the value of a Jodan PD8 account and the value of the PD8 account.
>
> Use this information to construct a graph showing the relationship between the value of the PD8 account and time.

This task comes in two distinct parts. The first part was intended to be an assignment in graphing the PD8 account in the phase plane: creating a graph of the rate of change of the account with respect to time (measured in dollars per year) as a function of the number of dollars in the account at that time. The second portion of this task was intended to be an assignment in constructing a graph of the value of the account as a function of time from the phase plane graph.

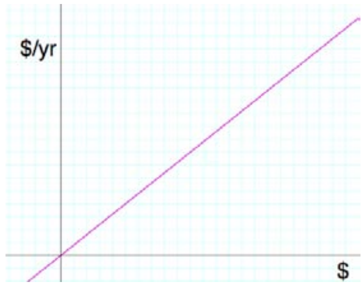


Figure 12. Phase plane graph of the PD8 account.

The original goal of this task was to introduce students to the type of "numerical analysis" reasoning used to describe the behavior of the Verhulst model in the previous chapter. That is, it introduces students to the idea that they could approximate the behavior of the continuously compounding PD8 account linearly over small intervals of time by fixing the dollar per year growth rate of the function at some point in time, imagining that the small amount of time passes and the function grows linearly at that rate, and then imagining a new value of the function and a correspondingly slightly higher growth rate.

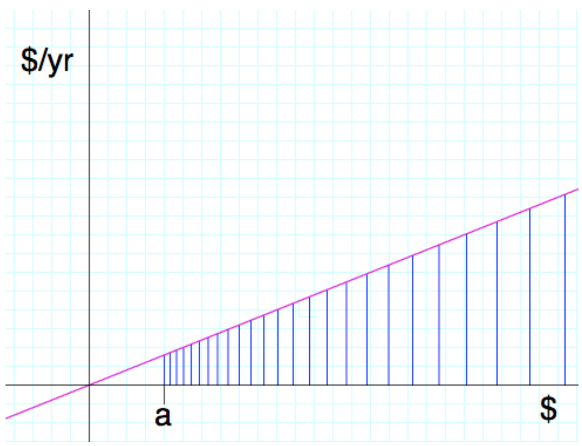


Figure 13. One possible representation of growth in the phase plane approximated in discrete time.

Figure 13 shows an example of this reasoning in the phase plane. Beginning from point 'a' the account has a certain value, and that value is associated with a particular rate shown by the height of the blue line at *a*. If we imagine fixing that rate for a short period of time, then the investment grows to the point where it reaches the next blue line, at which point we update the rate again. Over constant intervals of time, the changes in account balance (represented by the distance between the blue lines) is increasing due to the increasing rate of change (represented by the height of the blue lines). A caricature of the resulting graph of the account value over time is shown in Figure 14.

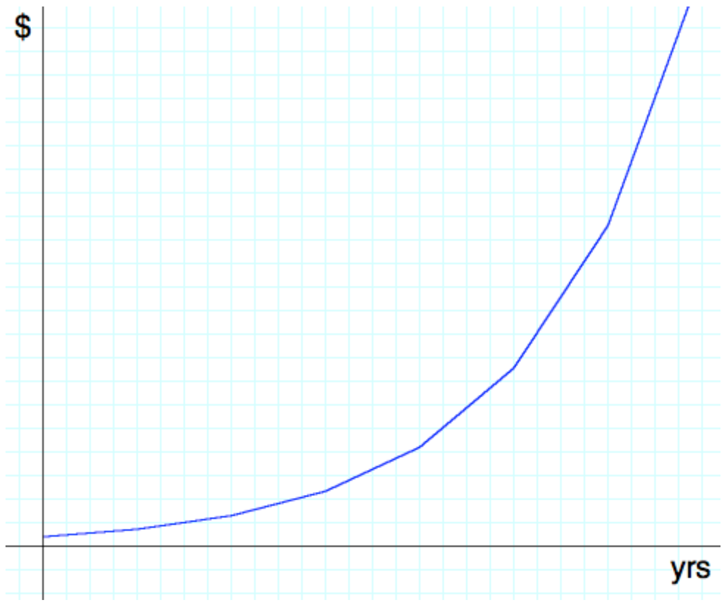


Figure 14. Approximation graph of the account over time, showing the rate being held constant for a period of time before being updated.

A key goal was that the students come to imagine that the next interval – being based in a slightly higher rate – would have a slightly higher increase in account value over the same period of time. The idea was that by imagining slightly larger increases in the change of the value of the account at one interval size, and then repeating this reasoning over multiple decreasing interval sizes, the student would arrive at a graph of the smooth exponential curve.

The entire instruction design of this task hinged on students becoming able to imagine approximating exponential growth (represented in phase plane and time series) with compound interest growth over arbitrarily small compounding intervals (represented in phase plane and time series), while at the same time recognizing that the compound interest model was only an approximation, and that a better approximation could always be attained.

I anticipated that this task would require an extraordinary amount of coordination on the part of the students. They would have to imagine time, rate, and account value changing all at once simultaneously on two graphs: one that depicted only rate and account value, and one that depicted only account value and time. The hypothesized solution to all this complexity was to slow the process of coordination down with the linear approximation method to reduce the number of simultaneous coordinations. In this method, the students would be asked to first, fix a small interval of time; second, chose a starting account value for that interval; third, imagine (and fix) a corresponding rate for that account value; fourth, by imagining that rate was fixed over that small interval of time, covary only time and account value; fifth find an ending account value based on that linear covariation.

The above steps are very closely tied to numerical values, so a final key step was to have students reason about the relationship between successive time intervals qualitatively, so that without the load of calculating numbers, the student could then imagine rate, account value, and time all changing simultaneously, and generate a qualitative image of what the "true" (non-approximate) graph must look like.

Malthus Model

At the turn of the 19th century, Thomas Malthus proposed this financial model as a model for predicting the population of the world. Using the properties and the behavior of the model, describe the good and bad points of using it as a population model.

This task was intended to be prefaced with the equation for the PD8 financial model in the students' own notation, which I anticipated as being something like y=.08x, with y defined as the rate of change of the account measured in dollars per year, and x being defined as the number of dollars in the account at any time. The purpose of this task was to create a transition from the financial context to the biological context, by discussion some of the reasons why that transition does not work perfectly. Part of the goal of this transition was that the students understand that the Malthus model works only as an approximation for the behavior of a mid size population. If a population is too large, it suffers from competition and starvation effects not covered by the model, while if the population is too small, the real valued population predicted by the Malthus model is a poor approximation of discrete population growth.

I hypothesized that the students would (and should) be uncomfortable with the idea of a population having a rate of change (and implied continuous growth), but rather would be more comfortable with images of their own experience of discrete population growth in discrete time: a family has no children for an interval of time, and then at the point of time corresponding to a "birth" the population is incremented by one. Part of the discussion I anticipated was building from this mundane image of personal population growth experience to an image of the behavior of the aggregate population – and corresponding aggregate rate – for which continuous approximation is more reasonable over fairly long periods of time. I anticipated that Derek and Tiffany's facility with transitioning between the individual level and the aggregate level would be critical for the subsequent task, in which they would begin to derive the Verhulst model from per-capita reasoning.

Per-capita Rate of Change

> Verhulst proposed the following modification to the Malthus model in order to make it more realistic. Instead of imagining that the per-capita rate of change was always constant, what would happen if that rate slowed down in response to population pressure, eventually becoming 0 for some population value called the carrying capacity? How would you write the model?

At the time of the design, I considered the final two tasks to be extremely nebulous. The realization of these tasks depended on how the students came to think of covariation, rate of change, and exponential growth as the result of all the previous teaching episodes and upon the notational schemes they developed for capturing them. I was extremely skeptical of my own ability to predict student mathematical understandings this far into the future. The final two tasks, relating to the Verhulst model, were then more a record of my own understanding of the Verhulst model, and my initial goals for the students' understanding of the Verhulst model, rather than any sort of statement of intended action. I anticipated that my questions, hypotheses, task, and even my goals would mutate by the time the teaching experiment would reach this point.

The goal for this task was to introduce a third graphical representation of the Malthus model, a graph of per-capita rate of change as a function of population (Figure 15), and to use that graph as a starting point to discuss the Verhulst model. Through this graph, I planned to have a discussion about how the constant per-capita rate of change of the Malthus model was unrealistic, and contrast with per-capita rate of change graph of the Verhuslt model. My goal for this second per-capita rate of change graph was simply that the students graph a decreasing function that reached zero at some population value (the carrying capacity).

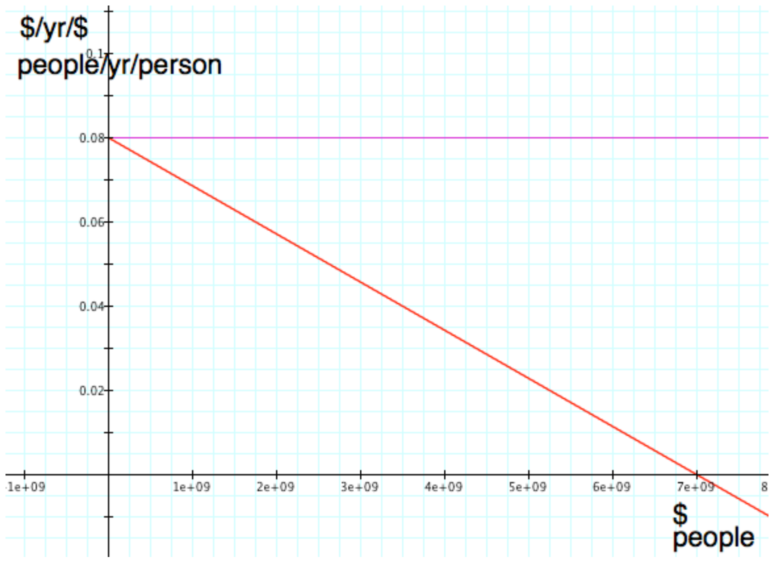


Figure 15. Per-capita rate of change as a function of population for the Malthus model (purple), and the Verhulst model with a carrying capacity of seven billion (red).

From this graph, I would introduce a similar graph of the Verhulst model (Figure 15), in the form of per-capita rate of change as linearly decreasing function of population (if the students did not create a linear function already). I anticipated that this function might need to be introduced because the linear function is a vestige of Verhulst's ad-hoc approach (Verhulst, 1977). Simply put, there is no biological justification to select a linear per-capita rate of change function over any other decreasing function.

$$\frac{\dot{y}}{y} = \left(\frac{-0.08}{7000000000} \right) y + 0.08 \qquad (17)$$

$$\dot{y} = .008\left(1 - \frac{y}{7000000000}\right)y \tag{18}$$

My final goal for this task was that the students generate a function for the per-capita rate of growth of the Verhulst model as a function of the population (Equation 17), and use that function to develop a function for the rate of growth of the Verhulst model (Equation 18). The functions written here are in my own notation, with y representing the population, $\dot{y}$ representing the rate of change of the population with respect to time, and $\frac{\dot{y}}{y}$ representing the per-capita rate of change of the population.

I anticipated that developing these equations with the students would require the same type of aggregation reasoning described in the per-capita exponential model section. That given the rate of growth for one individual in a population of identical individuals, the rate of growth of the aggregate population would be the sum of the individual rates of growth, resulting in the rate of growth for the population being the product of per-capita rate of change and population.

Phase Plane Analysis of the Verhulst Model

> Create a graph showing the relationship between the rate of change of the population with respect to time, and the population at that time.
>
> Use the graph you created to construct a graph showing the relationship the population and time.

This task comes in two parts. The first part – asking the students to graph the Verhulst model in the phase plane – was intended to be a repetition or reminder in order to pick up the conversation from the previous task. An example of one possible graph resulting from this discussion is shown in Figure 16.

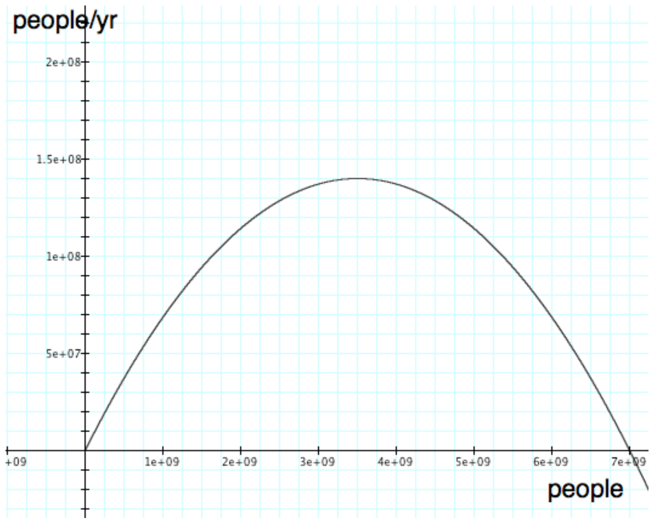


Figure 16. Phase plane graph of the Verhust model

The second task, to graph population as a function of time, was the real meat of the final task. My research goal here was to evaluate how the students understood phase plane dynamics by asking them to interpret a complex problem: one in which the behavior changed qualitatively as the population increased. My instructional goal was that the students develop the linear approximation approach described in the “Phase plane analysis” task above, as well as in the previous chapter. That is, I intended the students to create the population-time graph of the Verhulst model by imagining a piecewise linear approximation: fixing a small time interval, choosing a population, fixing a rate, imagining the population growing linearly at that rate over the small timer interval, and then finding a new population and a new rate. Given the difficulty of calculating values in this situation, I was particularly interested in stressing the qualitative behavior of the final function by comparing intervals: Is the previous rate larger or smaller than the current rate? Will the change in population of the next interval be bigger or smaller than the change in population over the previous interval?

The final component would be filling in the middle of each interval, recognizing that at all points along the linear approximation, the population and rate changed continuously, so that if the population reached the carrying capacity, the growth of the population would immediately stop, even if that carrying capacity occurred in the middle of a time interval.

One thing that I did not concern myself with was the asymptotic nature of the growth of the population. That is, I did not set as a goal that the students come to believe that the population growth will never reach the carrying capacity. I designed this task with the goal that a "general shape" of the logistic curve (Figure 17) would be "good enough" and I was not interested in introducing the mathematical equipment necessary to prove asymptotic behavior.

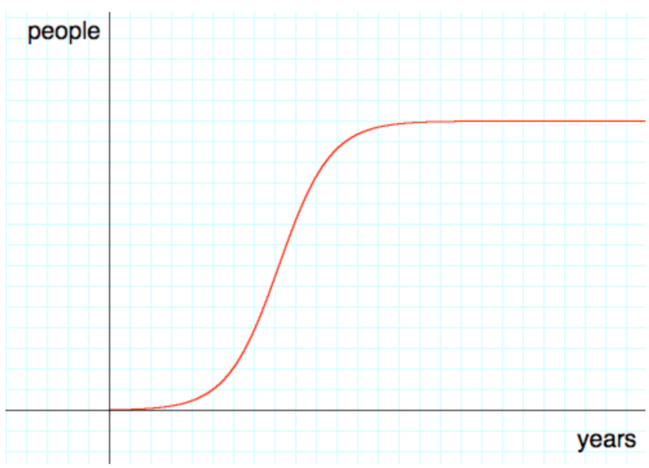


Figure 17. The qualitative behavior of a population over time in the Verhulst model.

A Day in The Life

Each day of the teaching experiment consisted of approximately the same routine: The day started with a teaching episode, in which I interviewed either Derek, Tiffany or both around one of the pre-prepared tasks in the sequence. If Pat was available to witness the teaching episode, then the teaching episode was followed immediately by a debriefing session, in which Pat and I shared thoughts on the teaching episode, and discussed what needed to be explored in the subsequent teaching episode. Following the debriefing session, I copied the video to a computer, and then created the first pass transcription of the video. Pat scanned his notes and emailed them to me. During this first pass transcription, I would look for the events that Pat and I discussed in the debriefing session and examine them in more detail. Finally, having re-watched the video as part of the transcribing process, I prepared the teaching episode for the next day, using the debriefing discussion, my models of the student's mathematics, and the pre-prepared original outline as guides.

Scheduling

As a result of Derek's extracurricular obligations, Derek was not able to attend every teaching episode at the originally planned time. Furthermore, as a result of changes to the schedule to accommodate Derek, Pat was not able to attend every teaching episode either. This resulted in a separation of the students early on in the teaching experiment. Later, Pat and I decided to separate the students deliberately. During our debriefing session following episode 9, we concluded that Derek and Tiffany thought so differently about the situations that they were asked to model that I couldn't adequately serve or study both students at the same time. Again, because this entailed meeting the students at times outside of the original schedule, Pat was not able to attend every teaching episode.

The initial scheduling of the teaching experiment was to run from March 2nd to March 12th. Unfortunately time ran out before either student was able to complete the entire sequence. Derek was kind enough to volunteer for a two followup interviews so that the full sequence could be completed as designed. Table 1 shows the final schedule of the teaching episodes.

Table 1
Schedule of Teaching Episodes

Date	Episode Number	Episode Name	Participants
3/02	E1D1T1P1	Simple Interest	Carlos, Derek, Tiffany, Pat
3/03	E2T2PT	Compound Interest	Carlos, Tiffany, Pat
3/04	E3D2	Compound Interest	Carlos, Derek
3/04	E4T3P3	Compound Interest	Carlos, Tiffany
3/05	E5D3T4P4	Compound Interest	Carlos, Derek, Tiffany, Pat
3/06	E6D4T5P5	Constant Per-capita Interest	Carlos, Derek, Tiffany, Pat
3/09	E7D5T6P6	Compound Interest on the Phase Plane	Carlos, Derek, Tiffany, Pat
3/10	E8T7P7	Compound Interest on the Phase Plane	Carlos, Tiffany, Pat
3/10	E9D6	Compound Interest on the Phase Plane	Carlos, Derek
3/11	E10T8	Limit of Compound Interest in the Phase Plane	Carlos, Tiffany
3/11	E11D7P8	Phase Plane Analysis	Carlos, Derek, Pat
3/12	E12T9	Phase Plane Analysis	Carlos, Tiffany
3/12	E13D8	The Malthus Model	Carlos, Derek
3/26	E14D9	The Food (Verhulst) Model	Carlos, Derek
3/30	E15D10	Logistic Growth	Carlos, Derek

CHAPTER 4

THE STORY OF TIFFANY

The purpose of this chapter is to provide a summary of Tiffany's work over the course of the teaching experiment, and the hypotheses that I developed about Tiffany's thinking during the teaching experiment, both on the spot during teaching episodes and between teaching episodes. A later chapter, "Retrospective Analysis" describes the model of Tiffany's thinking that I formed as a result of my retrospective analysis of the transcript and video of these events.

Overall, Tiffany participated in nine of the fifteen teaching episodes, covering the financial model portion of the original sequence. Tiffany participated in one teaching episode on simple interest, three teaching episodes on compound interest, one episode on constant per-capita interest, three episodes on compound interest in the phase plane, and one episode on phase plane analysis. Each episode lasted approximately 50 minutes.

Tiffany struggled with the material from the very beginning. During our debriefing sessions after each teaching experiment, Pat and I frequently commented on Tiffany's "chunky" thinking, essentially meaning that Tiffany thought about change discretely, although we did not formalize our meaning of "chunky" during the teaching experiment. Much of Tiffany's story is about characterizing the nature of "chunky" thinking and how this reasoning made the tasks difficult for her.

Simple Interest

In the first teaching episode, both Derek and Tiffany worked on the simple interest task below.

> Jodan bank uses a simple interest policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of eight percent of the initial investment per year. Create a function that gives the value of an EZ8 account at any time.

The simple interest task was initially implemented unchanged from my initial outline, but events took an unexpected turn as soon as the first task was implemented. In describing the model, the students interpreted the task in a way that I did not anticipate:

Excerpt 1 -- Episode 1, 00:02:29

1 Tiffany: Like if you put in a certain amount – I would say like ten dollars beginning

2 Carlos: OK

3 Tiffany: And then that the next year it should have grown, like I don't really know what eight percent of that is, but

4 Carlos: Eighty cents

5 Tiffany: Thank you. Like should have – So now in the next year she'd be like ten eighty cause there's like from-

6 Carlos: OK

7 Tiffany: Eight percent more and then the next year it's like, you know, whatever eight percent of ten-eighty is, so it should be doing something like that. That's how I look at it that's how I see it.

8 Carlos: [To Derek] What do you think?

9 Derek: Yes.

In line 7, Tiffany is describing a geometric compound interest model, an interpretation of the situation that Derek agrees with on line 9. Exponential growth had not yet been covered in the Algebra II class that the students were enrolled in, and I did not anticipate that the students would have had previous financial modeling experience, or that this task, as written, would be assimilated into those models. My response was to point to the original wording of the task.

Excerpt 2 -- Episode 1, 00:03:13

1 Carlos: So what does this eight percent of the initial investment per year mean?

2 Tiffany: Like the starting investment?

3 Derek: So it only goes eight percent of the very first each year. So if you put in ten, it only goes by eighty cents each year.

4 Tiffany: Oh, yeah like we were saying if you just put in ten dollars the next year, like you know how we were saying it would be ten-eighty? Well, it won't be per it won't be eight percent of ten-eighty. It'll still on … like only be eight percent of ten each year, instead of being eight percent of ten-eighty then eight percent of whatever ten-eighty is plus like, you know.[6]

In this excerpt, Derek and Tiffany are describing an image that is compatible with the idea of "rate" in linear modeling: that each discrete unit of time (a year), the account accumulates a fixed amount of money (80¢). Derek, (line 3) describes the situation where one begins with an initial investment of $10 as "it only goes by eighty cents each year."

[6] Ellipses (" … ") here indicate interrupted speech or disfluency. They do not indicate omitted text.

Tiffany (line 4) describes this as “eight percent of ten each year.” These descriptions are compatible with the idea of rate as the slope of a line, and in particular, are very similar to Confrey and Smith’s (1994) description of an additive linear rate: the repeated addition of 1 year, and the corresponding repeated addition of 80¢.

Based on this conversation, I asked Derek and Tiffany to imagine a customer of the bank named ‘Phil’ who invested $500 in an account of this type. I asked both students to write a function that gave the value of Phil’s account at any time (Figure 18)

$$500 + .08(500)(x)$$

Figure 18. Tiffany's function (formula) for the value of Phil's account at any time.

However, when I asked Tiffany to explain this function, Tiffany gave me a response that I overlooked.

Excerpt 3 -- Episode 1, 00:10:23

1 Tiffany: So I have the initial like amount that was put in then I have the rate that it's growing.

2 Carlos: And what are you circling there?

3 Tiffany: Uh, the point eight which is the rate, and then times the initial; so that together should be what like how much it's growing and then times the year.

In line 3, Tiffany clearly identifies “point eight” (referring to .08) as the rate, and identifies the product of .08 and the initial investment as a “how much” it’s growing, a different entity than the “rate.” I did not notice this at the time that Tiffany mentioned it, but I did address this later in the teaching episode when it reappeared. Following the creation of a function for Phil (Figure 18), I asked the students to create a function for any initial investment (Figure 19). Derek raised the concern that this new function would depend on two variables: the initial investment and the number of years, and asked how he could write that relationship using the notation q(). I began the function by writing q(x,n).

$$q(x,n) = n + .08(n)(x)$$

Figure 19. Tiffany's function for the value of an EZ8 (simple interest) account with any initial investment.

Following the successful creation of the general function, I revisited this issue of the meaning of the .08 in the function.

Excerpt 4 -- Episode 1 00:16:26

1 Carlos: What does that number really mean? What does the eight percent mean?

2 Tiffany: Umm …

3 Derek: It's the rate of change.

4 Tiffany: Thank you, yeah, that's kind of what I was going for.

In this excerpt, the students identify the 8% as the "rate of change," which is consistent with the financial interpretation Tiffany gave in Excerpt 1, but was not consistent with the idea of rate of change as the slope of a linear function, which was the meaning of "rate" the students had studied in class, and the meaning that I was depending on for future tasks. Much of the conversation that followed revolved around building the idea that .08*n*, rather than *.08* was the rate of change of this function, by looking at the change in an account over the course of one year. Tiffany (more talkative in this episode than Derek) summarized this discussion as:

Excerpt 5 -- Episode 1 00:18:57.01

1 Carlos: Is eight percent of five hundred dollars different from eight percent?

2 Tiffany: Yes, ‘cause eight percent maybe we don't know exactly. Like it could be eight percent of a hundred. It could be like eight percent of a million. We … we don't know exactly what eight percent we're looking at unless we have of a five hundred, or of this initial investment.

3 Carlos: If I were going to ask you how fast Fred's account is changing what would you tell me?

4 Tiffany: Eight percent of five hundred?

However, the students never developed a meaning for the 8% apart from the idea that “8% of something” measured change, which I think left them uncomfortable with the idea that 8% was not a rate, when they did not have any other meaning or term to assign to it. This conversation was left incomplete because I had not anticipated needing the idea of per-capita rate of change until much later in the teaching experiment, and did not think of a way to introduce the idea on the spot. The role that per-capita rate of change could have played in this discussion is described in the “Per-capita Rate of Change” section of retrospective analysis.

The final portion of this teaching episode was geared toward introducing the idea of proportionality into the idea of rate that we had developed from the previous conversation. Having established that after one year, an account holder would have increased the value of their account by $.08n$, where n is the initial investment, I turned the discussion towards developing a policy for what the bank should report if an account holder checked their balance during the year, rather than at year's end. The students established that in a fraction of a year, the account holder would earn that same fraction of $.08n$ dollars. At this point, I believed (prematurely) that the students were comfortable with the idea of rate as a proportional relationship between change in dollars and change in time, and after a little bit more questioning decided to move on to the next task.

Compound Interest

> The competing Yoi Trust has introduced a modification to Jodan's EZ8, which they call the YR8 account. Like the EZ8 account, the YR8 earns 8% of the initial investment per year. However, four times a year, Yoi Trust recalculates the "initial investment" of the YR8 account to include all the interest that the customer has earned up to that point.

Tiffany spent three teaching episodes working with this task. During the first two episodes that Tiffany worked on this task (E2T2P2 and E4T3P3), Derek was not present due to scheduling conflicts, but Derek rejoined for the final teaching episode in compound interest (E5D3T4P4). The full schedule is found in the Methods chapter.

The first teaching episode in this sequence, E2T2P2, consisted of my working with Tiffany to transition from a simple interest model to a compound interest model. I took some time at the beginning of the teaching episode to recap the simple interest model in the form of Phil, an imaginary customer who opened a simple interest account with $500. I planned this discussion with two intentions: The first being to remind Tiffany of the thinking that she had used previously and to get her back in the mindset that she had when first developing the simple interest model, which we would continue to use to build the compound interest model.

The second reason is that the compound interest task used a quarter of a year as the compounding interval, meaning that Tiffany was going to have to calculate simple interest values in fractional years. In the first episode, E1D1T1P1, the bulk of the conversation occurred around year 0 and year 1, and the interest earned in one year [in Phil's case, .08(500) or 40 dollars]. It was only at the very end of episode E1D1T1P1 that we discussed the interest earned in a smaller increment of time (months and days). The compound interest task, with the compounding occurring every quarter of a year, relied heavily on the ability to calculate changes in the account over quarter-year intervals.

After ensuring Tiffany's ability to calculate changes in the account over quarter-year intervals, I introduced the compounding task to Tiffany, and the remainder of the teaching episode consisted of contrasting Phil's simple interest account to the account of a new customer, Patricia, who invested in a compound interest account. These contrasts were discussed in the context of calculating the value of each account in quarter year increments. In what follows, Tiffany explains how to calculate values of Patricia's compound interest account.

Excerpt 6 -- Episode 2, 00:10:43.598

1 Tiffany: You would divide up the year into four parts. And then you look at like the next, like one of the quarters of the year and see: ok, after from zero to this quarter of a year, it's changed this much. And then you could take that number and do the uh rate thing. And then we'd have to look at the quarter of the year again and see the change there. And then you'd have to redo uh the rate and— it's a little bit more complicated than, um, yesterday.

For the first quarter, Tiffany observed that the both accounts would be identical. And used the calculation shown in Figure 20 for both Phil and Patricia's account.

$$500 + 0.08\,(500)\,\frac{3}{12}$$

Figure 20. Tiffany's solution to the value of Phil's account after a quarter of a year.

Tiffany also observed that Patricia's account would behave differently from Phil's account in the second quarter, specifically that in Patricia's case the bank would use the value at the end of the first quarter ($510), rather than the value of the initial investment ($500) to calculate Patricia's earned interest by the end of the second quarter (Figure 21).

$$0.08\,(500)\,\frac{1}{2} + 500$$

$$510 + 0.08\,(510)\,\frac{1}{4}$$

Figure 21. Tiffany's calculation for the value of Phil's account at the end of half a year (blue) contrasted with Tiffany's calculation for the value of Patricia's account at the end of the second quarter (green).

The remainder of the teaching episode involved rewriting Tiffany's calculations in various forms. Specifically, she used the distributive property to rewrite the calculations for Patricia's account as seen in Figure 22. The greatest hurdles here were that Tiffany did not remember the distributive property, and did not "see" 500 as the product of 500 and 1; however the details of that discussion are not relevant to Tiffany's image of covariation, and are not included here.

$$500(1 + .08/4)$$
$$510\left(1 + \frac{.08}{4}\right)$$

Figure 22. Tiffany's rewritten calculations for the value of Patricia's account after one quarter (top) and two quarters (bottom) after some instruction in the distributive property.

After generating these forms, I asked Tiffany to substitute her calculation of 510 (Figure 22 top) for the 510 in her calculation of the second quarter (Figure 22 bottom). This resulted in the calculation shown in Figure 23.

$$500(1+.08/4)\left(1+\frac{.08}{4}\right)$$

Figure 23. Tiffany's calculation for the value of Patricia's account at the end of the second quarter.

The episode ended with the unprompted observation by Tiffany that the final form of the calculation (Figure 23) could be written as a square, and based on the pattern of multiplications, Tiffany made the prediction that the value of the account at the end of the third quarter would involve a cube.

During this teaching episode, I discovered an interesting phenomenon; specifically, that in calculating values for a quarter of a year, Tiffany preferred to write "3/12" rather than "1/4" for the amount of time that had passed. An example can be seen Figure 20, above, which is Tiffany's solution to the question. "If Phil checked his balance at the end of the first quarter of the year what would he see?" Note that despite my phrasing of the question using the word "quarter," Tiffany elected to use 3/12 as the time.

I suspected that Tiffany's choice of 3/12 rather than 1/4 was because she was imagining cutting up the year into months, and then counting the number of months. With some prompting, Tiffany acknowledged that a 1/4 could be written in place of the 3/12. Her explanation of this equivalence was particularly illuminating.

Excerpt 7 -- Episode 2, 00:16:45

1 Tiffany: So if you do a quarter of a year or you just take three 12ths – literally, a quarter of a year – then you should get the same thing.

Particularly interesting in this portion is Tiffany's seeming insistence in the above excerpt that "three 12ths" is more literally a quarter of a year than one fourth. Tiffany followed up by explaining that each twelfth was a month.

Excerpt 8 -- Episode 2, 00:17:25

1 Tiffany: It's like how many months, like twelve months in a year and we're looking at three of those months.

In Excerpt 8, Tiffany's speech reveals that although she was saying "twelfths" in Excerpt 7, she was imagining months. It is here that I formulated my first hypothesis regarding Tiffany's covariational thinking: that Tiffany always imagined variation as occurring in unit sized chunks. This model explained why Tiffany, when pressed to think about fractional years, was more comfortable with the idea of three whole "one month" units, rather than $1/4^{th}$ of a "one year" unit. Over the course of the teaching experiment, Pat and

I came to refer to Tiffany's covaration as "chunky," a label that was retained later, as my model of Tiffany's thinking changed.

E4T3P3 – Compound Interest

The following day, Tiffany and I continued our exploration of the YR8 compound interest account. This time, Tiffany showed a marked preference for imagining the behavior of the function in quarter-of-a-year sized increments.

Excerpt 9 -- Episode 4, 00:01:36

1 Tiffany: So and then that's the end of that quarter and then we take that number look at the next quarter.

2 Carlos: If Phil goes and checks his account every day

3 Tiffany: The account the money is growing a little bit each day. So his account … he's gonna have a tiny bit more money each day in his account then he did the day before.

4 Carlos: OK.

5 Tiffany: So something like that, umm, so that at the end of the quarter he'll have a certain amount and then they'll use that number.

In the above excerpt, Tiffany initially demonstrates chunky thinking, with "a quarter" serving as the unit size of a chunk. She describes the value of each quarter as being calculated from the value of the previous quarter, as if nothing happens during the chunk itself. When I asked Tiffany about the day-to-day behavior of the function, she described it as "growing a little bit each day" but immediately and without prompting returned to her quarter-by-quarter description.

In the next portion of the teaching episode, Tiffany explained her geometric calculations from her previous teaching episode (Figure 24), and repeated her prediction that for the value of the compound interest account at the end of three quarters, the factor would be cubed.

$$500\left(1+\frac{0.08}{4}\right)^2$$

Figure 24. Tiffany's calculation for the value of Patricia's compound interest account at the end of the second quarter.

At this point, Pat asked Tiffany why the calculations included division by 4. Tiffany's response revealed a great deal about the nature of her chunky reasoning.

Excerpt 10 -- Episode 4, 00:06:10

1 Tiffany: If you don't divide it by four you have the whole year that's gone by. You've got your full amount that you've earned. And then if you put the divide it by four, you only have that one section of the year. You're looking at the one quarter.

In this excerpt, Tiffany describes a one-year chunk of time, and a corresponding one year chunk of interest. She then describes a process of cutting up the one-year chunk of time into four pieces, and imagining selecting one piece of the four, resulting in a chunk the size of "one quarter." Implied in this reasoning is that the chunk of interest undergoes a similar process, resulting in the interest chunk corresponding to the one quarter [of a year] chunk. This line of reasoning is superficially similar to the line of reasoning that I myself was using, and at that time I was unaware of the differences except for a vague sense of unease. It was not until retrospective analysis that I was able to sufficiently articulate the differences between our two perspectives.

Following this exchange, I put Tiffany to the task of calculating the value of Patricia's account for various non-quarterly amounts of time, with the intention of preparing her for constructing a piecewise linear function. My goal for this section was that she begin to think about separating each time given into a number of quarters and a remainder, and the corresponding value of the account at that number of quarters (calculated geometrically) and the amount of interest earned over the remainder (calculated linearly). In this section, Tiffany worked with four time values: 0.1 years, 0.3 years, 4 months, and 0.6 years.

$$500 + .08(500)\boxed{.1}$$

Figure 25. Tiffany's calculation for the value of Patricia's compound interest account at 0.1 years. The box around .1 was added later.

0.1 years required only that Tiffany use the linear calculation, and Tiffany succeeded at this task without difficulty (Figure 25). However Tiffany wanted to use the same approach for .3 years, a difficulty that occurred because Tiffany did not have a sense for how long .3 years was, specifically, that it was longer than a quarter of a year. I will not discuss .3 in detail, because our discussion around .6 years covers the same ideas in more depth.

Tiffany however, was comfortable with the idea that a quarter was three months (E2T2P2), and my second attempt at approaching this problem was to task Tiffany with finding the value of Patricia's account after four months. Tiffany's first inclination was to use $500 + .08(500)(1/4)$ to find the amount earned after one quarter, and

$510+.08(510)(1/12)$ to find the amount earned over the remaining month, and add those amounts together to find the amount earned after four months. That is, Tiffany saw each calculation as calculating the amount Patricia earned over that time period, rather than the total amount Patricia had at the end of some period of time. With some inquiry about the meaning of the parts in her calculation, she identified 510 as the amount earned in one quarter, $.08(510)$ as "the rate" and $.08(510)(1/12)$ as the amount earned over the remaining month. Tiffany revised her answer to $510+.08(510)(1/12)$.

The most interesting case was the final task in this portion of the teaching episode, specifically, Tiffany's initial reaction to the task of finding the value of Patricia's account after .6 years.

Excerpt 11 -- Episode 4, 00:16:18.668

1 Carlos: The value of Patricia's account after point six years.

2 Tiffany: Point six years? Uh kay. Well we would have to figure out how much of the year that really is.

Tiffany was clearly uncomfortable with the idea of 0.6 years (line 2, underline is used to indicate vocal emphasis), and stated that she needs to figure out how much of a year 0.6 years "really is." Tiffany quickly clarifies:

Excerpt 12, Episode 4, 00:16:42.074

1 Tiffany: Like we could figure out how many quarters that is or something, and then so the same type of thing. Like if it's just point six, I don't know right now. I don't know how much of a year – oops – that really is,

Here Tiffany explained that finding out what point six is involves finding out how many of a smaller unit of time 0.6 years takes up. She proposed finding out how many quarters 0.6 years is, which is a necessary step to find out how many factors of (1+.08/4) are needed, but Tiffany also proposes finding the meaning of 0.6 as a number of months or days:

Excerpt 13 -- Episode 4, 00:17:3.5140

1 Tiffany: we'd need to know how many months – if you wanna look at months – si … point six of a year is. So maybe not necessarily days, but like kinda the same things. But we do it with months maybe?

This discussion led directly to my second model of Tiffany's mathematics, which we (Pat and I) had come to call "chunky." Specifically, in E2T2P2, I hypothesized that Tiffany's thinking was based in thinking in whole number "chunks." In this episode, I hypothesized that as a result, she didn't really have a sense for how large decimal or fractional amounts were unless she could rephrase them in terms of a larger whole number of smaller units. In the 0.3 years case, Tiffany did not see 0.3 years as being larger than 1/4 of a year, in part because of the different representations (I believe that she would recognize 0.3 as being larger than 0.25). In the same sense, Tiffany really didn't have a sense of how big 0.6 years really was compared to other fractional amounts of years, like quarters.

In order to make sense of the fractions of years that were being tossed around, Tiffany wanted to re-imagine 0.6 years as a whole number of quarters or months or days, just as her initial inclination when presented with a quarter of a year was to imagine it as 3 months (E2T2P2). Thus it was not only Tiffany's covariational system that operated in whole units, but her entire rational number system as well. 3/12 was not 3/12 of one unit, but 3 of some units that were called "twelfths." 0.6 years did not have a sufficiently small unit for Tiffany to give it a sense of scale by this method.

My response was to propose that Tiffany rewrite ".6" as ".5+.1" and this helped Tiffany because she could interpret the .5 as two quarters which enabled us to proceed. With that 'two quarters,' Tiffany was able to remember the value of Patricia's account after two quarters, and using the four months calculation as a guide, generated the calculation shown in Figure 26.

$$520.2 + 0.08(5202)\underline{.1}$$

Figure 26. Tiffany's calculation for the value of Patricia's account after .6 years.

In the final portion of this teaching episode I tasked Tiffany with finding the value of Patricia's account after *x* years, essentially, asking Tiffany to construct a function predicting the value of Patricia's account. Tiffany proposed using the same method that she used for four months and for .6 years: breaking the time into a number of quarters and a remainder, and finding the account value after that number of quarters, and the interest over the remainder. Using this method, Tiffany constructed functions for the first three quarters (Figure 27) with one major hiccup; Tiffany did not originally make a distinction between *x* as the number of years that had passed, and the remainder. This distinction was made by asking Tiffany to study her previous work with four months, and Tiffany making the observation that although four months had passed, the number used in her calculation was 1/12, not 4/12. Repeating this reasoning let Tiffany to the form (x-1/4) in

Figure 10 middle. For each formula, I asked Tiffany which values of x “were ok to use” and she wrote the domains of the pieces using the notation that I have copied into the figure caption.

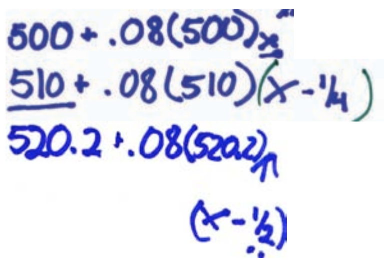


Figure 27. Tiffany's function pieces for the value of Patricia's account after x years. Top: On a domain of 0<x<1/4. Middle: On a domain of 1/4<x<1/2. Bottom: On a domain of 1/2<x<3/4.

Generalizing these functions required that Tiffany substitute her geometric calculation for each quarterly value for the calculated numbers used in the functions above, resulting in Tiffany’s final function forms. Once Tiffany began making these substitutions, she was able to generalize the form and find the function piece for the sixth quarter without finding all the functions in between (Figure 28).

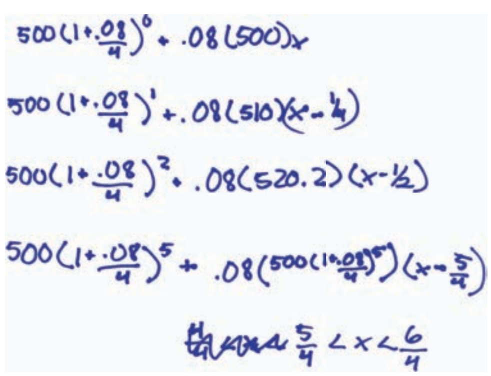


Figure 28. Tiffany's function pieces for the first quarter (top), second quarter (middle top), third quarter (middle bottom) and sixth quarter with domain (bottom).

Domains are omitted from the first three pieces because she had written them previously.

E5D3T4P4

In this teaching episode, the two students were reunited. This was the final teaching episode in which the students worked on a function for the value of a compound interest account in terms of time. I began by asking the students to recap the work that they had done so far. I'm including a portion of Derek's response here as well because the contrast to Tiffany's reveals a lot about the way in which she was thinking. In this excerpt, the students are explaining the (x-1/2) portion of Tiffany's function piece for the third quarter (Figure 28 middle bottom).

Excerpt 14 -- Episode 5, 00:07:58

1 Derek: So the x minus one half it's for … Like if you have something more than half a year but less than three quarters, so that when you get the first half then you have to figure out the leftovers by getting how ever many years put in for x or how much of the year. Then subtract the first half cause you already figured it out over here [pointing to $500\left(1+\frac{.08}{4}\right)^2$] so then when you multiply by the rate you wanna get that much of the rate. Then you add it on.

Derek's explanation of the function is in terms of a value somewhere within the domain of the function. If you're given a value of x larger than "half a year" but less than three quarters, then the amount in the account is the value of the account after half a year, and then adding the amount earned in the remaining (x-1/2) years. It is important to note that Derek does all of his reasoning while using *years* as a unit. In the next excerpt, Tiffany explains the same portion of the compound interest function.

Excerpt 15 -- Episode 5, 00:09:07

1 Tiffany: And we'd be like: ok, well, we wanna know more than just half the year. We wanna know uh half the year and a month. So if we take that half the year and we subtract it by half well, we'll have that month, and like. I think like kinda something like that.

Tiffany's explanation, while similar in structure of splitting time into half a year and a remainder, differs in a crucial way. While Derek is reasoning entirely with a unit of one year (as a continuous quantity), Tiffany is mixing units of years and months (discrete quantities). This method of mixing units keeps the "chunks" that Tiffany is working with to sizes that she can judge and compare. This is necessary in order for Tiffany to judge if the amount of time is suitable for the domain of the function. Derek's explanation is rooted in the idea that any real number of years can be compared to half a year and three quarters of a year based on its place in a continuous number line, without a mixing of units.

Hoping to connect with Tiffany, I gave my own interpretation of the function entirely in language of months, rather than years. As an instructor, this had reached the point where this task was taking too much time, and I hoped to build their ways of thinking in continuous quantities with graph work later in the teaching episode. As a researcher, I hypothesized that Tiffany's "chunky" reasoning would be problematic in the future, but I was also curious to see what Tiffany could build from it. It seemed to me at this time that I did not have the resources to change Tiffany's thinking at this moment, since the equations the students had to work with were well suited to thinking in chunks. From both perspectives, I was eager to move on to the graphing phase of Episode 5.

The second phase of Episode 5 involved predicting and generating graphs of Phil's simple interest account and Patricia's compound interest account. First, I asked the students to predict what a graph of Phil's simple interest account would look like. Tiffany predicted a line. The students entered their equation for Phil's account into graphing calculator and saw that it was indeed a line. When asked, the students each gave their interpretations of the graph.

Excerpt 16 -- Episode 5, 00:17:19

1 Tiffany: This graph shows that his … It starts at five hundred, because at zero years it starts at five hundred. And then it increases little by little every year, and not even every year but every like parts of it. Just little by little.

At the time of this teaching episode, I took Tiffany's description as evidence that Tiffany was beginning to reason continuously as a result of the graph. In retrospect, this passage, is a subtle example of Tiffany thinking in chunks, and the contrast with Derek's explanation has a greater significance than I understood during the episode.

Next, I asked the students to predict the behavior of Patricia's account. Tiffany was unsure what Patricia's account would look like, and Derek sketched the graph shown below.

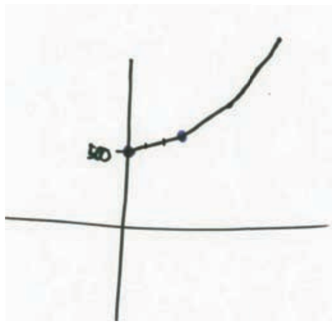


Figure 29. Derek's graph of Patricia's account over time.

I then asked the students to graph the function for the value of Patricia's account with respect to time, which the students accomplished with only minor hiccups that did not reveal much upon later analysis. Upon completing their graph over a domain of the first three quarters (Figure 30), both students correctly asserted that the lines composing the graph were of increasing steepness, even though there was no visual evidence that the graph was anything other than a line.

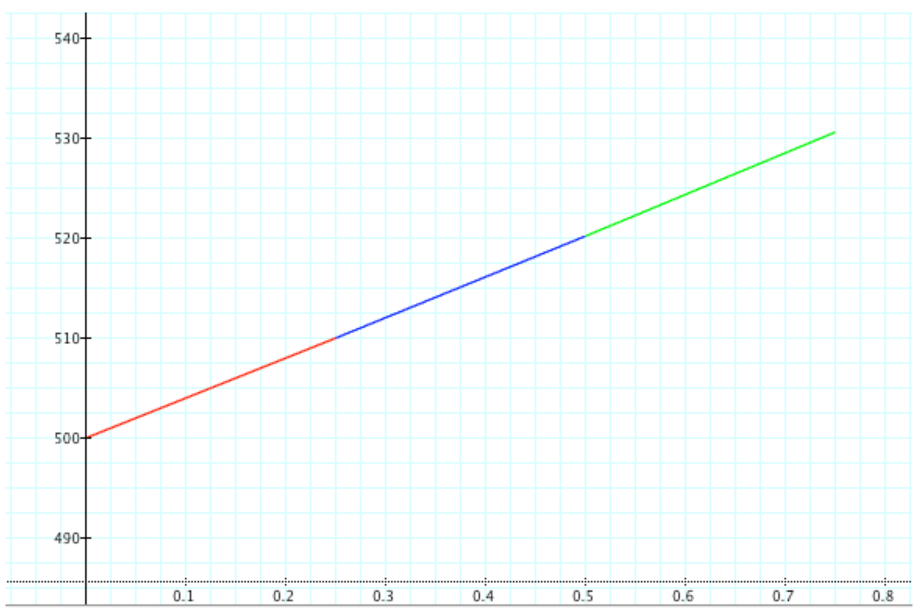


Figure 30. Derek and Tiffany's graph of Patricia's account over time.

In the final phase of this teaching episode, I worked with the students on a few quick bank policies that Pat and I developed during the previous debriefing session. The policies were designed so that each account would have the same value on the quarters, but differ in value between quarters. The goal of these policies was to help the students reason continuously by showing them an animation of each policy being graphed in time. Although the animations were successful in eliciting continuous speech from Tiffany, they had no lasting impact on the thinking of either student.

Constant Per-capita Interest

> The Savings Company (SayCo) also competes with Yoi Trust and Jodan. SayCo's PD8 account policy is as follows: if you have one dollar in your bank account, you earn interest at a rate of 8 cents per year. For each additional dollar, your interest increases by another 8 cents per year. If you have fractions of a dollar in your account, your interest increases by the same fraction, so 50 cents earns interest at 4 cents per year. Here is SayCo's new feature: At any moment you earn interest, SayCo adds it to your account balance; every time your account balance changes, SayCo pays interest on the new balance and calculates a new growth rate. Why is SayCo's PD8 the most popular account?

Tiffany's reaction to this problem was that it is the "most complicated" of the policies that they had seen so far, which is true. However, the complexity in this policy is largely artificial, a product of multiple perspectives that the task was designed to embody, as detailed in the methods section.

Excerpt 17 -- Episode 6, 00:1:34

1 Tiffany: It … it was … Instead of by saying like an initial investment of like any like amount of money, it's a dollar. Like, if you have a

> dollar in you account you earn eight cents for it, and then if you have another dollar you earn more eight cents. It's, like, more money, and it's just more. Like, oh! Fraction of a dollar you only earn the fraction of it.

Tiffany saw the problem as complicated because the tracking of each individual dollar complicated the problem for her. In Tiffany's description in Excerpt 17, Tiffany describes each dollar as being associated with an amount (eight cents), rather than a rate (8 cents per year), and Tiffany envisioned the accumulation of all those 8 cents as happening by accumulating all the dollars. Other than the issue of time (and thus rate) being dropped from the discussion, this is actually a considerable portion of the interpretation that I intended. My intention was to build the idea that, as a result of these accumulations (and by applying distributivity), rate is always proportional to amount.

However, Tiffany's difficulty with this accumulation began to make clear to me that I had neglected a very important idea in the previous tasks: that it was not only this policy which could be thought of from the point of view of individual dollars, but all of the previous tasks. Tiffany's understanding of rate in the context of a financial model seemed to be "the amount earned at the end of a time unit." This way of thinking was sufficient for Tiffany to get by during simple interest and compound interest, but was not sufficient in the context of a rate that was always changing: I was concerned that with a rate that was always changing, there would be no unit of time that she could look ahead to.

As a result of Tiffany's interpretation, this task left me in the position of having to teach the idea of a per-capita perspective and the idea of a dynamic rate simultaneously. I chose instead to introduce these ideas one at a time: beginning with a per-capita perspective and constant rate of change. This meant that much of this teaching episode was spent re-teaching simple interest from a per-capita perspective.

Teaching per-capita simple interest involved building two ideas: The proportionality of the rate to the initial investment, and the proportionality of the amount of interest earned to the amount of time that had passed. I began with proportionality of rate to initial investment, building the idea one dollar at a time, and building proportionality of interest to time within each initial investment scenario. Below, Tiffany is describing a simple interest account with an initial investment of two dollars.

Excerpt 18 -- Episode 6, 00:7:22

1 Tiffany: That every year if you just keep the two dollars in your account you have sixteen cents per year. Like every year they'll add one more, like, sixteen cents, I think. Yeah, that how this account works? Yeah. OK.

2 Carlos: So umm does that mean that umm for a year, It's two dollars every day? Two dollars, two dollars, two dollars, two dollars, two dollars, and then on December thirty-first it's two dollars and sixteen cents?

3 Derek: No, like it

4 Tiffany: Depends on how they tell you. Like if you … like we were talking about before. If they're going to tell you what's in between – how much it's grown in between – or if they're just going to tell you the starting and ending of the year.

5 Carlos: So if they tell you how much it grows in between, what'd it be like?

6 Tiffany: Small. Umm. It would be like one … maybe one 12th of your sixteen cents if you're looking at it by month or one 365ths if you're looking at it per day.

Tiffany initially describes the behavior of the simple interest account in chunks: For every year sized chunk of time that accumulates, the account accumulates a chunk of 16 cents. Another interesting aspect of Tiffany's reasoning in this excerpt is that Tiffany doesn't assume proportionality of change in cents to change in time as being an inherent part of the meaning of 16 cents per year. Instead, she says that what goes on during the year depends on the bank's policy. Once it is established that the account value is always changing, Tiffany breaks up the year chunk into smaller month or day chunks.

The second portion of this teaching episode involved transitioning from the simple interest model to the originally intended exponential model by introducing compounding through "SayCo's new feature." The student's initial interpretations of SayCo's new feature are in Excerpt 19.

Excerpt 19 -- Episode 6, 00:24:14.94

1 Derek: So every time you get up to one more cent the rate changes

2 Tiffany: -think so, so, every time it's changed then they take that new number and-. Wow, they do that a lot then.

3 Carlos: What do you mean by they do that a lot?

4 Tiffany: Because if it changes by, lets say by the three point … the three dollars – we'll just use … that's easy. So in a year it's eight cents so you're now take that eight cents to three oh eight.

5 Derek: Well it's not by the end of the year.

Here Tiffany and Derek have two very different interpretations of SayCo's new feature. Derek sees the rate as changing discretely, but in unequal compounding intervals: whenever one cent is earned: in order for a change to be noticeable (on a statement) would have to change by at least one cent, so one cent is the minimum change. Tiffany interprets the problem as being compounded annually: I interpreted this as Tiffany looking at the changes as occurring in one year "chunks," so updates can only occur at the end of the chunk because Tiffany is not imagining changes within the chunk.

Because my original intention was to build the relationship of rate proportional to amount, and specifically not continuous compounding, my response was to take time out of the discussion at this point. I transitioned to highlighting the independence of the situation from what had occurred before: I asked the students to imagine that each student had 37 dollars in their own personal account, but that these accounts had been opened at different times. After some discussion about how the rate policy was calculated, the students concluded that both accounts were currently earning money at the same rate of .08*37 dollars per year, independent of how many years the account had been opened.

From here, I asked the students to construct a function predicting the rate of growth of the account from the value of the account (Figure 31).

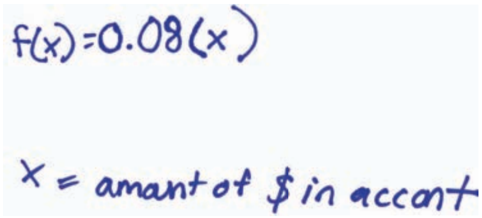


Figure 31. Tiffany's function for predicting the rate of growth of the account.

In the debriefing, Pat brought to my attention a problem that I had until that point considered largely solved, which was that there were too many meanings of rate floating around the discussion. The phrase "growth rate" had four different meanings. I viewed growth rate as a number of dollars per year. Tiffany viewed "growth rate" as the amount earned in a unit time. Derek viewed growth rate as "how fast," and Pat viewed growth rate as ambiguous between "interest rate" and dollar per year rate.

To contrast Tiffany's and my meanings: my meaning of growth rate, at the time of Episode 6, meant that the growth rate over one day would be 16 cents per year, in the same sense that one could travel 70 mi/hr for one second. Tiffany's idea of "growth rate" means that in order to find the growth rate after a day, she must divide 16 cents by 365 to find the amount earned in one day.

Derek's understanding of rate is best contrasted with Pat's. Pat distinguished between two rates: an "interest rate" of 8% per year, and a dollars per year rate of .08*N (if N is the account balance). Derek did not make this distinction, regarding both growth rate and interest rate as synonymous, with both terms referring ambiguously to the number .08 and to .08*N dollars per year.

Based on the debriefing, in which Pat pointed out that the students and I had different meanings of "growth rate," we decided that the focus of the next teaching episode would be on clarifying the meaning of growth rate that I intended.

Compound Interest on the Phase Plane

> Previously: Yoi Trust one-upped Jodan's EZ8 account, which was a simple interest account, with their YR8 account, by adding earned interest to the principal investment at the end of every 3 months (4 times per year).
>
> For the Yoi YR8 account, create a graph that relates the value of the account at any time expressed as a number of dollars and the amount of interest earned during a very short period after that time expressed as a number of dollars per year.

In total, Tiffany worked on this task in three teaching episodes: E7D5T6P6, E8T7P7, and E10T8. She worked with Derek only in the first of these three teaching episodes, Episode 7. Derek had another scheduling conflict preventing him from attending episode 8 at the regularly scheduled time, and Pat and I decided to use the opportunity to separate the students for the remainder of the teaching experiment.

This task was originally designed to introduce the idea of phase plane to the students, by having students create a phase plane graph for a compound interest account. In later episodes this task was used for that purpose. However the task as presented to the students (above) differed from the task as initially designed (methods). The changes were made in reaction to the previous teaching episode and debriefing session, in which it became clear that there were entirely too many meanings of rate floating around. My intention was to clarify my own usage of rate to the students by describing, essentially, the average rate of change of the function (difference quotient) over a small interval. However, the students were not comfortable with two ideas that proved critical to this understanding of rate: the idea of equivalent rates, and the idea of a very small interval of time. As a result, the entirety of the teaching episode E7D5T6P6 was a discussion around interpreting the question being asked. A third issue was that the idea of ratio never came up at any point in the discussion, which made the discussion of equivalent rates very difficult.

Excerpt 20 -- Episode 7, 00:6:29

1	Derek:	I think it's like, you know, the each quarter. Like during one quarter you're earning the same amount of interest for that time, so this like that short period would be that quarter cause it's all the same, see. It doesn't change 'til the next quarter.

Derek introduced the idea of equivalent rates with this comment, in which he essentially describes that because the rate of change (dollars per year) of the compound interest function over a small interval would be the same as the rate of change over the entire quarter, because the rate is constant for a quarter (until the compounding). With further poking, Derek described a model of equivalent rates to which we returned frequently during the course of the discussion. In the excerpt below, Derek is responding to my question “What’s the same for a quarter?”

Excerpt 21 -- Episode 7, 00:7:46

1 Derek: Well not the interest earned, the interest … You'll get the same interest back by the end of the quarter. Like they'll all add up to the same, so-

Derek described an equivalence of rates based on accumulation: that two rates defined over different time intervals are the same rate if as the same amount of time passes, both rates accumulate (add up to) the same amount of money. Derek's thinking did not change from this position during the course of the teaching episode. However, Tiffany was having difficulty with Derek's extremely brief explanation, and I put the students to the task of filling in the details more carefully.

Excerpt 22 -- Episode 7, 00:11:00

1 Carlos: So, let's think about it this way. Umm. Over that quarter of a year I earn a certain amount of interest and a certain amount of time has passed, and the time has passed is a quarter of a year. Now if I look at, umm, say, oh, even an hour in that quarter of a year I earn a certain amount of interest, and a certain amount of time has passed. An hour has passed. Now, I understand … I understood [Derek] as saying that, in some way, even though they earn a different amount of money and a different amount of time has passed, those are the same in some way. So, how are they the same?

What followed was a discussion of how to calculate the interest earned over an hour from knowing the interest earned over a quarter (divide time and interest by the

2190 hours in a quarter), and how to calculate the interest earned over a quarter from knowing the interest earned over an hour (multiply both time and interest by 2190). This style of discussion reflected the reasoning they used to convert speeds to different units at the very beginning of the academic year. However, I was unable to bring the conversation home.

Excerpt 23 -- Episode 7, 00:16:39

1	Carlos:	In two thousand one hundred and ninety hours. So I've figured out the amount of interest that I've earned in a quarter, because I've multiplied the interest by two thousand one hundred and ninety, and I've multiplied the number of hours by two thousand one hundred and ninety.
2	Tiffany:	Mhm. OK.
3	Carlos:	So what stayed the same? When I did that?

The students were unable to answer this question adequately, and at the time, so was I. The solution is simply that the ratio of the interest earned and the time elapsed stays the same; regardless of the size of the time elapsed. However, with my strategy of imitating the style of unit conversion reasoning that the students had used previously in the year, the idea of ‘ratio’ never came up. By discussing equivalent rates in terms of changing units such as hours and years, the equality of ratios superficially does not hold, and so did not occur to me.

Pat's approach in this instance was much the same as mine, he asked the students about the speed of a car as an analogy, also invoking the reasoning the students had used at the beginning of the year.

Excerpt 24 -- Episode 7, 00:21:14.

1 Pat: Umm, if I'm going sixty-five miles per hour what does that mean?

2 Tiffany: That in one hour you've gone, you should have gone sixty-five miles.

3 Pat: All right, now can … can I express that speed in feet per second?

This exchange, and the discussion that followed did very little to alter Tiffany's thinking in the long run, but it became one of the most important passages for my retrospective analysis, because it provides clear examples of Tiffany's "chunky" thinking in a context that Tiffany was familiar and comfortable with. Note Excerpt 24 line 2, in which Tiffany describes 65 miles per hour has having gone one hour, and having gone sixty five miles: Sixty-five miles per hour describes a one hour "chunk" of time and a corresponding chunk of sixty five miles. At the time I did not place a particular amount of importance on the details of Tiffany's chunky thinking about speed, since it confirmed the model I already had. It was only in retrospective analysis that I was able to use this conversation to fill out the details of the operations of Tiffany's chunky thinking.

In the 65 miles per hour conversation, Pat, Derek and Tiffany established that 95 feet per second was the same speed as 65 miles per hour, and Derek made the analogy to the interest rate example, but the nature of 'same' still eluded the students. Pat's discussion of converting miles per hour to feet per second took us all even further from the idea of equivalent ratios (Excerpt 25).

By building up the accumulation model in the speed example, the students were able to successfully make the connection between equivalent rates in a manner that did not depend on ratio (Excerpt 25).

Excerpt 25 -- Episode 7, 00:30:49

1 Carlos: I'm still going the same speed. I'm still going the same distance in the same amount of time.

2 Derek: It's just uh, if you have a line and for 45 miles and an hour and you want per minute, you just divide it into 60ths. So you have a 60th of 45. And then a 60th there's sixty of them so they add up to 45 miles.

3 Carlos: Ok so [Tiffany] that … that explanation that [Derek] just gave about the line and dividing it up, umm how would you explain the account in the same way?

4 Tiffany: Umm the interest should still be the same so, so if we looked at the interest and we just divided it up it should still add up to the same thing like in the end. If … if you like divided and just look at a part of it, and in the end it should still be the same kinda thing.

In lines 2 and 4, Derek and Tiffany describe the same model of equivalent rates: a model based on partitioning and accumulation. In each explanation, one rate was calculated form the other rate by partition, and the rates were equivalent because the same process could be reverse by accumulation: 45 miles per hour can converted to miles per minute by partitioning hours into minutes and the length 45 miles into the same number of partitions, each of length 0.75 miles. The rates are equivalent because the process can be reversed by accumulation: as minutes accumulate to form an hour, the 0.75 mile chunks accumulate to form 45 miles.

The remainder of the teaching episode involved re-introducing the students to the task, and to the tools that they would have to complete the task: their work from previous episodes. After reintroducing them to the graphs and functions that they had produced for the YR8 compound interest account there was no time remaining for working with the task itself, and so the task was postponed to the following episodes.

Due to another conflict of schedules, the students were unable to meet together for the next teaching episode. In the debriefing, Pat and I decided to make the most of this opportunity by separating the students for the remainder of the teaching experiment. It became increasingly clear over the course of the teaching experiment that the students had very different ways of reasoning, and the decision to separate them was so that I could adapt to build off each way of reasoning directly, rather than putting each student on hold as I addressed the other.

E8T7P7

The following day, Tiffany and I continued with the task of graphing a compound interest account in the phase plane. I worked with Tiffany as she constructed several graphs of various compound interest accounts in the phase plane using parametric reasoning. This teaching episode opened with Tiffany providing a description of the YR8 account graph that she had constructed in Episode 5 (Figure 30).

The next portion of the teaching episode was centered around the goal of plotting a single point on the phase plane trajectory, specifically, the point that would occur when the account was first opened. For this trajectory, we used Patricia's account, as we had in the past: an initial investment of $500, growing at 8% per year, compounded quarterly. For this problem Tiffany identified two pieces of information that she would need: the number of dollars at time 0, which she easily identified as 500, and the number of dollars per year at time zero, which turned out to be more problematic.

Excerpt 26 -- Episode 8, 00:06:19

1 Tiffany: Alright well for that point, we would know like that it … You'd know the number of dollars in the account, but we since it's zero years we wouldn't necessarily know the, uh, interest because interest is over time. So we'd know, ok, like, let's say it's five hundred dollars. I think it would be five hundred dollars just with what she has, but I don't really know where … like, how we would do the, um, interest part of that.

Here Tiffany's understanding of rate as an amount caused problems for her. Her understanding of rate as an amount and an associated unit of time means that she needed a unit of time to have passed in order to calculate the amount. In the case of the time the initial investment is made, no time has passed, so Tiffany did not have a time unit to base her rate-amount on. This difficulty was resolved by once again asking Tiffany to make an analogy to a situation she was more familiar with: driving a car.

Excerpt 27 -- Episode 8, -- 00:18:36

1 Carlos: Umm, driving a car at … So, let's imagine that I was driving a car, umm, at 45 miles an hour and I did that for fifteen minutes. When those fifteen minutes were up, umm, I speeded up to sixty-five miles an hour. How fast was I driving in the first minute?

2 Tiffany: Umm, you would have to convert the forty-five miles per hour into miles per minute. Sorry. Or feet would it be ffff-, No cause you want to know how many miles right you've gone in the first minute?

3 Carlos: No, I wanted to know how fast I was driving.

4 Tiffany: Oh how fast! Oh! [laugh] Umm well you're still driving forty-five miles an hour in the first minute.

5 Carlos: But I only did it for a minute, does that make a difference?

6 Tiffany: No.

Tiffany's initial reaction to my question of how fast I was driving in the first minute was to want to calculate the distance that I had traveled in a minute, again showing Tiffany's understanding of rate as an amount of change associated with a unit time. However, once I clarified that this was not the meaning of "how fast" that I was looking for, Tiffany drew on her driving and sociolinguistic experience of "miles per hour" to give me an answer that was independent of a time chunk.

Based on this analogy, Tiffany was able to extend the idea of the amount earned in a quarter to dollars per quarter over an hour (Excerpt 28).

Excerpt 28 -- Episode 8, 00:22:1

1	Carlos:	In the first hour how many dollars per quarter am I earning?
2	Tiffany:	Umm. Ten dollars per quarter
3	Carlos:	Why?
4	Tiffany:	Because it's still it's the same rate? It's the same, umm. Kinda like the same speed, like what we were talking about before. That's the growing rate, and that's what the bank is giving you.

In the excerpt above Tiffany developed a meaning of dollars per quarter that is independent of the time passed. Rather than being an amount of dollars earned in a quarter, Tiffany uses the car example as an analogy, so that she imagines that the “speed” that the account changes at is not affected by the size of her time chunk. This independence of rate from elapsed time enabled Tiffany to plot the first point of her phase plane graph at the coordinates ($500, $40/yr).

Tiffany quickly identified a second point corresponding to one hour after the initial investment as having slightly more money, and the same rate. Similarly for three minutes after that, and so on. Tiffany predicted that the overall graph would be a line continuing horizontally until it reached $510, which is when the compounding occurred.

Once Tiffany was able to find the phase plane trajectory point by point, she quickly generalized: the next step would continue horizontally until it reached $520.2, the value at which the next compounding event occurred, and so on. This led Tiffany to create the graph shown in Figure 32 fairly rapidly.

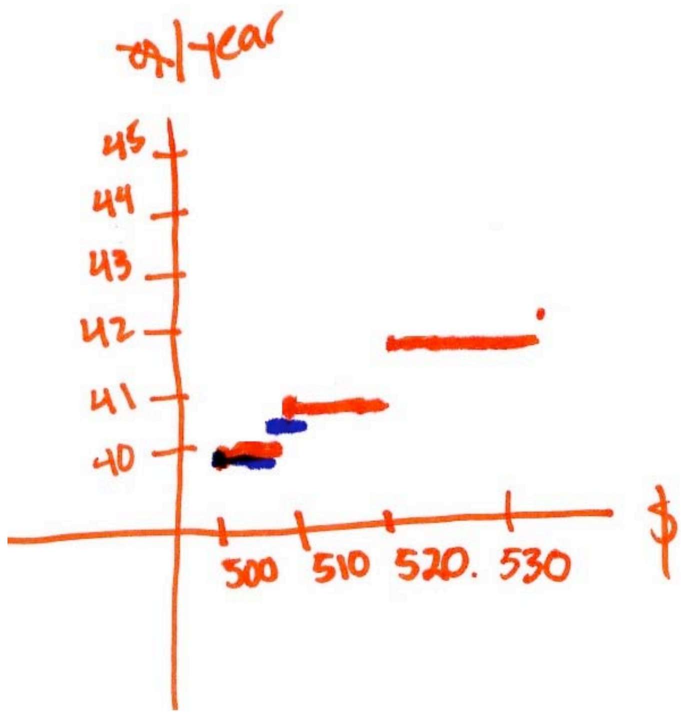


Figure 32. Tiffany's phase plane trajectories of a \$500 investment growing a 8% per year compounded quarterly (orange) and five times a year (blue).

Following Tiffany's constructing the graph in Figure 32, I questioned her on her interpretation of the graph that she just created. I asked her to interpret a point on the graph, which she explained as telling her the amount of money in the account, and how much the account would earn in a year if there was no compounding.

I next asked Tiffany to explain how to find the height of a step. Tiffany came to the generalization (needed later for taking a limit) that the height of a step was 0.08 times the value of the account at the beginning of a step (when the account was compounded).

In the final portion of the teaching episode, I introduced a new character, Heather, who shopped around at different banks with different compounding policies. Using

Heather, I asked Tiffany to predict the graphs that would be made for smaller and smaller compounding intervals. In each case, Tiffany predicted that the steps would be shorter and lower, as shown in Figure 32.

E10T8 – Limit of Compound Interest in the Phase Plane

This teaching episode was a continuation of the task that I had assigned Tiffany of graphing compound interest accounts in the phase plane. My goal for this task was to establish that as the compounding interval decreased, the relationship between the rate of the account and the value of the account became proportional, and to connect this result with the per-capita interest PD8 account function f(x)=.08x that Tiffany and Derek had created in Episode 6.

I opened this teaching episode by asking Tiffany to explain the graph she had created in her previous teaching episode (Figure 32). Tiffany successfully identified the orange graph as a graph of Patricia’s compounded quarterly account, the blue account as a graph of Heather’s compounded five times a year account, and both graphs as being graphs of dollars per year with respect to dollars in the account. In the course of this discussion, Tiffany described the idea of rate of change much more fluently than she had in the past.

Excerpt 29 -- Episode 10, 00:03:08

1 Tiffany: So but if we were looking at this, and we just wanted to look at: ‘kay I have five hundred and ten dollars in my account, and I don't want to worry about cutting up the year into quarters. Just want to know, OK, if I hade five hundred and ten dollars in my account.

Well I could be earning forty-one point six dollars a year. Without worrying about picking the year saying, "OK, this is one quarter; now it's changing. Now it's changing again," everything.

Tiffany’s description of 41.6 dollars per year in Excerpt 29 differs greatly from the descriptions of rates that she had given in the past. Rather than describing a rate as an amount corresponding to a particular unit of time, here Tiffany is describing a hypothetical amount: the 41.6 dollars is the amount the account would earn under the hypothetical condition that the account was allowed to accumulate money without compounding for a full year.

The next task that I proposed to Tiffany was to create a function that relates the value of Patricia’s account at the end of a compounding period to the rate of change of the function for the following compounding period. Tiffany responded that this function could be used for Heather’s “compounded five times a year” account, or any account with any compounding interval, so long as the input was always the value of the account at a compounding event.

Figure 33. Tiffany's function for finding the rate of change of an account given the account’s value.

From here, I asked Tiffany to imagine an account compounded every second, which I (not Tiffany) used as an approximation of compounding continuously. Tiffany successfully predicted that the phase plane graph of an account compounded every second would resemble a straight line; however, I was unable to determine how she came to this conclusion, particularly whether it was based in recognizing the form of y=0.08x, some pattern in the phase plane step functions she had created to date, or reasoning based on the relationship between the rate and the account value. Tiffany answered tautologically that the graph of her function y=.08x would be whatever graph you got if you graphed of the function y=0.08x: Essentially, the graph is a line because when you graph the function, you get a line, suggesting that she recognized the form of the y=0.08x, but I cannot discount other possibilities with certainty. I concluded this section with a Graphing Calculator animation showing the phase plane step functions converging on a line as the compounding interval was shortened.

The next section of the graph involved comparing the properties of the limit account, which I called “Heather’s approximation” to the properties of the PD8 constant per-capita rate account. I opened this discussion by asking Tiffany to graph the relationship between a distance and time of a car driving 65 miles per hour. In the discussion of the graph, I described the graph as showing a relationship between changing number of miles and a changing number of hours, and Tiffany identified the constant rate of the relationship as “sixty five miles per hour.” Using this reasoning as a basis, Tiffany successfully identified the rate of change of the function for Heather’s approximation (the function y=.08x relating rate of change and account value) as .08 dollars per year per dollar.

Tiffany remembered the PD8 account (Episode 6), and also identified some differences between Heather's compounding every second scenario and the PD8 per-capita interest account.

Excerpt 30 -- Episode 10, 00:40:44

1 Tiffany: That's [Heather's account] compounding every second because it actually has a compounding interval. Like, the bank is always changing it every second. This one [PD8 account] is just like not changing. Not really changing. It's just saying, "OK you can use however much is in your account to figure out how much you're earning per year."

In this excerpt, Tiffany comes very close to the interpretation of the PD8 account I originally had intended: that the PD8 account is a simple statement of the proportionality between rate and amount, without any of the machinery of imagining linear growth over discrete intervals. However, Tiffany states this insight in a way that is inaccurate, stating that the [rate of the] PD8 account is "not really changing." That is, although Tiffany imagined the proportional relationship between rate and amount described in the PD8 account policy, she did not imagine it in a context where the account value was changing continuously in time.

Tiffany's final task was to graph the value of Heather's compounded-every-second account over time. Tiffany created this graph second by second and point by point, rather than making a graph of the overall shape, she focused on showing the relationship between each second to the previous second, resulting in an entirely discontinuous graph, that nonetheless had the shape of the exponential curve, as Tiffany imagined each change in account value (Tiffany's chunky rate) increasing every second.

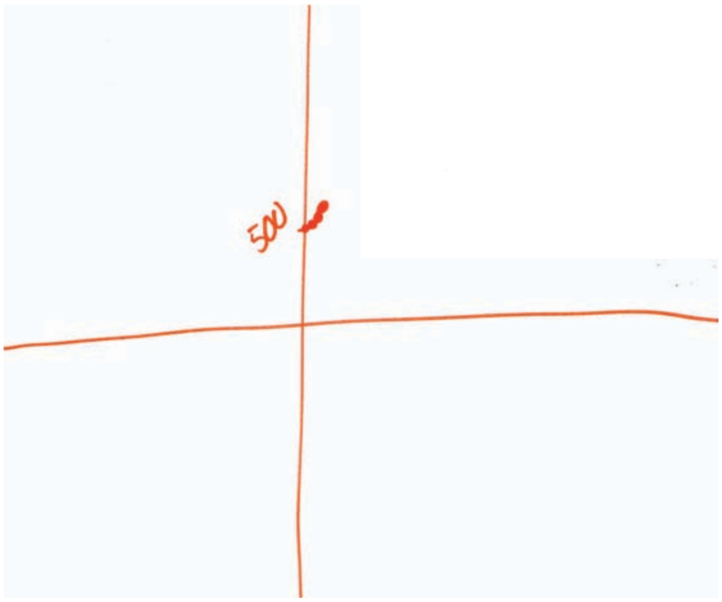


Figure 34. Tiffany's graph of the value of an account over time beginning with \$500 and compounded every second. Each dot on the graph represents a second. Tiffany was very explicit that the dots were not connected.

Phase Plane Analysis

Tiffany's final teaching episode began with a review of the PD8 (constant per-capita growth rate) policy and her work from episode 6.

Excerpt 31 -- Episode 12, 00:0:15

1 Carlos: What can you tell me about this account policy?

2 Tiffany: This account policy is, like, where they do eight cents for every dollar, and then they change like every time you get a new … your balance changes. So every time, like, that money changes you get

… they recalculate your growth rate based on what like that changed to. So it changes rapidly a lot.

It seems from Excerpt 31 that Tiffany is beginning to imagine that the policy is compounding continuously, or at least discretely but frequently, as for instance in the example of compounding every second. However when asked for clarification, Tiffany gave a description of the account compounding annually.

Excerpt 32 -- Episode 12, 00:0:42.05

1 Carlos: So what do you mean by every time your balance changes?

2 Tiffany: Like, um, every time. Let's say you put in, let's say, we put in a dollar, and then we earn interest on that dollar and it's a dollar oh eight. Well instead of taking that dollar to earn the interest, they take the dollar oh eight to make the interest. It's like kinda move up.

In this example, Tiffany begins by imagining an initial investment of a dollar, and then imagines the “the interest” (the rate-amount) to change when the investment reaches “a dollar oh eight”--the amount that the account would be worth if no compounding had happened during the year. I interpreted this as Tiffany imagining the account changing in one year chunks. After stepping through the model in hundredth of a second increments, Tiffany explained that the rate would change, and so the function would not be 1.08 at the end of a year, but more. In the course of this discussion, I stepped Tiffany though a hundredth of a second, two hundredths of a second, and a full second.

Excerpt 33 -- Episode 12, 00:4:45.364

1 Carlos: We had like a hundredth of a second, ummm, and then they did compounding updating thing. Another hundredth of a second, they did the compounding and updating thing. And then one second they did this compounding and updating thing. Does that mean that they only compounded three times?

2 Tiffany: No

3 Carlos: So

4 Tiffany: ‘Cause they still like used the hundredth of a second in this. So it could have been a hundred and … and three times, or it could have been more.

5 Tiffany: Just because you could have gone by maybe not a hundredth of a second you could have gone by like a thousandth of a second. They're still compounding it.

During the teaching experiment, I interpreted Tiffany as saying in lines 4 and 5 that the bank was always compounding, so that it is compounding every hundredth of a second but also every thousandth of a second. What I did not notice until retrospective analysis, is that what Tiffany is describing in lines 4 and 5 is not continuous compounding, but rather that the policy Tiffany has understood does not specify a definite compounding interval. The bank could have compounded every hundredth of a second, in which case it compounded 103 times (line 4), or the bank could have compounded every thousandth of a second. This ambiguity in Tiffany’s interpretation of the policy created a striking result later in the teaching episode. For now, I was content to move on, imagining that Tiffany was compounding continuously.

I next asked Tiffany to explain the graph that she had made previously (Figure 34) of the value of the account over time, with a point at each of three seconds. Tiffany described how each change in the value of the account would be bigger than the previous change in the value of the account.

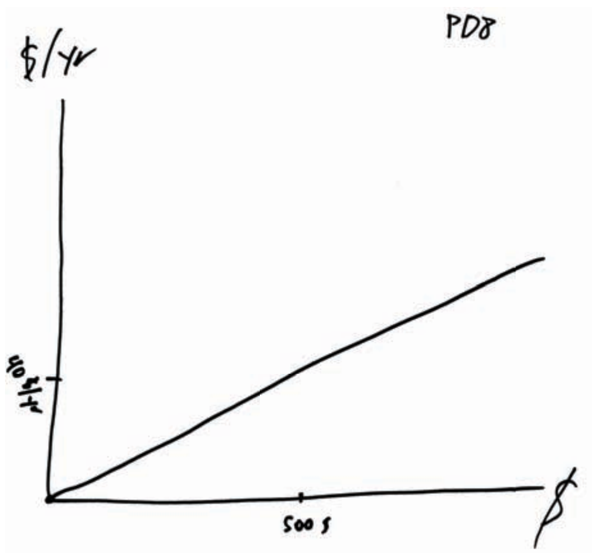


Figure 35. My version of the phase plane graph of the PD8 account. I drew this during this teaching episode as a blown up version of a much smaller sketch Tiffany drew during episode 10.

I next presented Tiffany with a phase plane graph of the PD8 account. I asked Tiffany to place her finger on the x-axis of this graph (Figure 35). And to move her finger to represent how the value of the account was changing as time changes. As Tiffany did this she moved her finger along the horizontal axis continuously, but did not show any sign of speeding her finger up. As I questioned Tiffany further on what the graph told Tiffany about how to move her finger, Tiffany gave an explanation very similar to the numerical approximation method I described in the methods section.

Excerpt 34 -- Episode 12, 00:18:54

1 Tiffany: Like, um, this point here tells me that at five hundred dollars I'll be earning forty dollars per year. So if I wanted to say look a year later I would have to move forty over. But if we're not looking necessarily at a year, then it just tells … Like it kinda helps with that. Kinda tells you how much to move by.

In this excerpt, Tiffany fixes a time chunk, and uses rate as amount earned over a unit time to calculate how much to move her finger by. In this example Tiffany used years as her time chunk, but she acknowledged that other units of time would be possible.

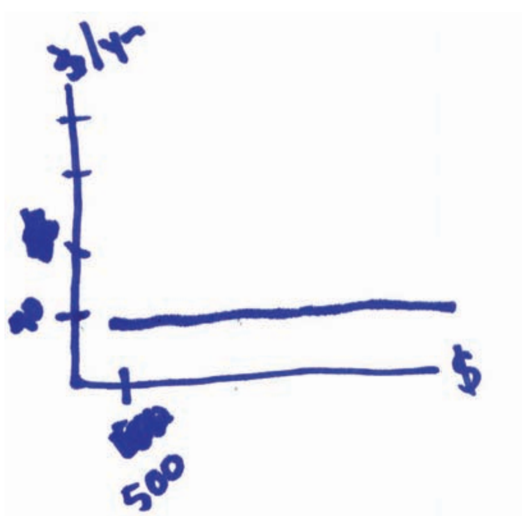


Figure 36. Tiffany's phase plane trajectory for a simple interest account beginning with 500 dollars at time 0.

In the course of this discussion, Tiffany brought up again the idea that the rate was the amount that the account would earn in a year if the compounding process did not occur.

Tiffany’s idea of the rate of the account being how much you earned without compounding led me to try having Tiffany study the behavior of a simple interest account in the phase plane before finishing the PD8 account analysis. Tiffany created a phase plane graph for the simple interest account (Figure 36).

Tiffany and I used this graph in a collaborative effort to produce a graph of the account value over time. Tiffany placed fingers on the x and y axes of the phase plane graph to represent the value of the account and the rate of growth of the account at a particular time. I placed my fingers on the dollar and year axes of a second empty graph to represent the values of those quantities at the same time. Tiffany moved her fingers and gave directions to me to tell me how to move my fingers. Specifically, she repeatedly instructed me to move my time finger one quarter, and my dollar value finger ten dollars. She moved her own dollar finger ten dollars at a time, and kept her rate finger at 40 dollars per year, which she used to calculate the 10 dollar finger movements.

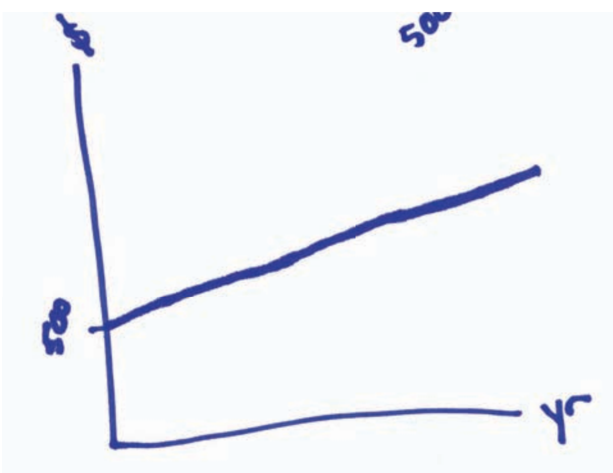


Figure 37. Tiffany's graph of the simple interest account over time, based on her phase plane trajectory.

Continuing in this way, Tiffany and I sketched out the graph of a simple interest account, which she drew as a line. (Figure 37). I then asked Tiffany to do the same process with the PD8 account phase plane graph. For the PD8 account, I asked Tiffany to direct our movement second by second. Tiffany established that over each second, we should each move our dollar amount fingers a little bit more than what we moved them over the previous second. I then asked Tiffany what happened during a second.

Excerpt 35 -- Episode 12, 00:37:36

1	Carlos:	Now tell me what happened during that second.
2	Tiffany:	That second during that second you earned a little bit of money so now they're gonna recalculate and stuff for your new-
3	Carlos:	OK,
4	Tiffany:	-rate.
5	Carlos:	but that's not what I meant.

Tiffany responded by treating the entire second as a single chunk, without any events occurring within the second, either linear growth or compounding. In order to describe what happened within a second, I asked Tiffany to explain the process in tenth-of-a-second intervals. Tiffany explained that each tenth of a second the change in dollar amount would be more than the previous tenth of a second. However this placed us in an endless loop. In order to discuss what happened during a tenth of a second, we repeated the process in increments of a hundredth of a second, and a thousandth of a second, and so on, without any resolution.

Excerpt 36 -- Episode 12, 00:40:02

1 Carlos: What about, um, thousandths of seconds?

2 Tiffany: Should be more jumpy cause you'd see like the tiny bit, and then a little bit more than that, and a little bit more than that, little bit more little more.

3 Carlos: Would you be able to see the jumpiness?

4 Tiffany: No unless you're like really zoomed in.

Anticipating that Tiffany would apply her thousandth of a second reasoning, I asked Tiffany to graph the first three seconds of the PD8 account over time. Tiffany behaved very differently than I expected, graphing three points (one for each second before she ran out of space for the third second) instead of a smooth curve.

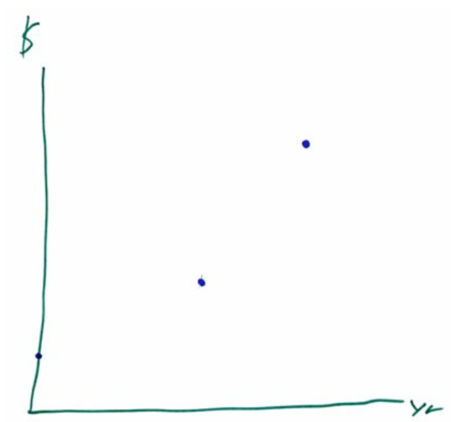


Figure 38. Tiffany's graph of the value of the PD8 account over time for the first two seconds.

When I asked Tiffany how to fill in what goes on in between points, Tiffany repeated the process on a smaller scale, filling in point by point, and describing a process of using each point to find the next point. She described the overall function as “jagged.”

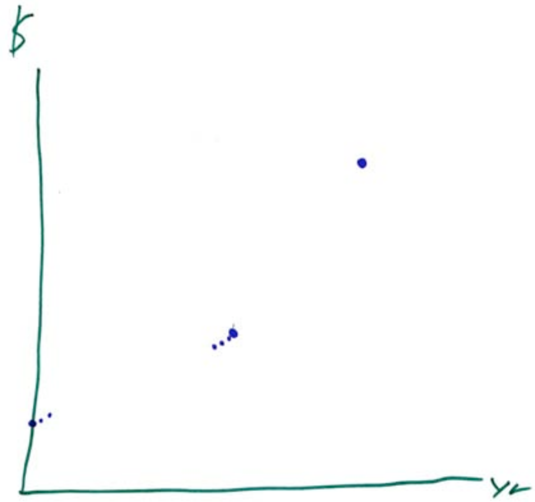


Figure 39. Tiffany fills in the first second of the account point by point.

In Figure 39, Tiffany fills in the first second of the account with two sequences of points, one sequence near 0 seconds, and one sequence near one second. Within each of the two sequences each change in height is higher than the one before, but the two sequences are not placed so that the sequences will connect, giving no sense of the over all shape of the curve. Tiffany does not have an overall shape of the curve either.

Excerpt 37 -- Episode 12, 00:43:56

1 Carlos: So now if you looked so small you couldn't see the jaggies what would it look like?

2 Tiffany: It would look like a l … just kinda like a … Don't know why I keep closing this. Kinda like if you couldn't see the jaggies it would look like that like a line.

3 Carlos: OK.

4 Tiffany: Solid.

5 Carlos: So by a solid line you mean like a straight line?

6 Tiffany: Yes.

7 Carlos: So it would be like a straight line from here to here [gestures between the first and second points] and then a straight line from here to here [gestures between the second and third points]?

8 Tiffany: Yes that's what I … I think that's what it would look like.

In Excerpt 37 I make the mistake of suggesting a shape to the overall graph (a line) before asking Tiffany to sketch it out herself, however the Tiffany agrees with my interpretation of her statement as piecewise linear, and she is very confident in her final statement in line 8. In retrospect, this opens one of three possibilities. The first, and most likely possibility is that Tiffany simply didn't imagine anything going on in-between points until the issue was forced. The remaining possibilities are that either Tiffany thought the graph was piecewise linear all along, or Tiffany imagined non-lineararity between the points, but was influenced by my linear suggestion, in which case Tiffany did not have a very strong image of what the graph would look like.

Tiffany's two sequences of points (Figure 39) are (very slightly) non-linear, with each change in height between points being higher than the previous change in height. This would seem to indicate the third possibility. The idea of a non-linear but uncertain graph is accentuated by the independence of Tiffany's two sequences of points: they don't line up so that one sequence feeds into another in a smooth curve.

However, there is also evidence for the second possibility – that Tiffany sees linearity between each second. The first issue concerns the confidence with which Tiffany agrees with my suggestion of lines. The second piece of evidence comes from earlier in the episode, specifically Excerpts 33 and 34, in which Tiffany explains that the compounding interval you choose is flexible. In Excerpt 33 Tiffany describes the behavior in hundredth second intervals, and then describes compounding every thousandth of a second as an alternative. In Excerpt 34, Tiffany describes compounding every year as an option (the account would earn 40 dollars in a year, indicating linear growth over the year), but that it could be compounded more frequently.

If this is the case, then for each observational increment, Tiffany is imagining piecewise linear compound interest growth. This leads to the phenomenon that the compounding interval is based on the observational unit, rather than being set by bank policy. If Tiffany is asked to imagine the PD8 account in one-second increments, then she imagines the account compounding every second. If Tiffany imagines the account in one-tenth-of-a-second increments, then she imagines the account being compounded every tenth of a second. This seems to be consistent with Tiffany's approach of finding each point from the previous point, and her assertion that the graph would be "jagged," and it is consistent with the first sequence of points not leading toward the point at second 2. However it is not consistent with the second sequence of points leading toward the point at second 2.

Tiffany's understanding of the exponential is particularly interesting in that it appears that she achieved my written goal for the teaching experiment: In using compounding to graph the value of the PD8 account over time, Tiffany demonstrated that she learned to interpret phase plane graphs using linear approximations. However, it was not until Tiffany reached this point that I discovered that the goal I originally committed myself to was not the goal that I held now. I now wanted Tiffany to see the graph of the PD8 account over time as a smooth curve.

I worked briefly with Tiffany on finding the point at one half of a second, but none of that conversation helped to resolve the ambiguity surrounding Tiffany's imagined graph. At this point, we ran out of time for the teaching experiment, and this was Tiffany's last teaching episode. However, I was satisfied that Tiffany's understanding of compound interest growth was fairly sophisticated, and her understanding of continuous compounding was functional: No matter what Tiffany imagined going on between the points, a curve would emerge from her simply graphing a long enough sequence of points close enough together.

CHAPTER 5

THE STORY OF DEREK

Similarly to the previous chapter, this chapter gives an account of Derek's participation in the teaching experiment. Derek participated in ten of the fifteen teaching episodes: one on simple interest, two on compound interest, one on constant per-capita interest, three on compound interest in the phase plane, one on phase plane analysis, and one on the Malthus model. In addition to these eight teaching episodes, Derek also volunteered to participate in two additional interviews two weeks later: one on deriving the Verhulst model, and one on logistic growth.

Overall, Derek was a very difficult student to model. Derek did a lot of his work in his head, and answered questions quickly, and generally in the way that I intended those questions to be answered. Derek's quick and easy responses made it difficult to ascertain how he might be thinking about the tasks that I assigned to him in two ways. Because Derek responded to my questions so quickly, it was difficult to give him tasks that would slow him down and force him to externalize his reasoning. Because Derek frequently responds with answers that were (from my point of view) correct, it was difficult to find pretexts to push Derek further into a problem that had already been solved. This problem was greatest in the early teaching episodes, It was not until later in the teaching experiment, when Derek began to work with more challenging tasks, that I was able to begin forming models of Derek's mathematics.

The problem of modeling Derek's mathematics is exacerbated by a paucity of data. In the teaching episodes in which Derek and Tiffany worked together, Tiffany struggled with the material more than Derek did. Much of my dialogue in these two student episodes is geared toward assisting Tiffany, and Derek would make few contributions during these episodes. This is a large part of the reason why Pat and I decided to separate Tiffany and Derek later in the teaching experiment.

Simple Interest

In the first teaching episode (E1D1T1P1), both Derek and Tiffany worked together on the simple interest task below.

> Jodan bank uses a simple interest policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of eight percent of the initial investment per year. Create a function that gives the value of an EZ8 account at any time.

Tiffany and I did the majority of the talking in this teaching episode. A sense of the sequence of events that occurred can be found in her chapter. In this section, I will remark only on the events of the episode unique to Derek.

The episode began with a discussion of the meaning of the text that I had given them. Nine minutes into the episode, I asked Derek and Tiffany to imagine a customer of the bank named 'Phil' who invested $500 in an account of this type. I asked both students to write a function that gave the value of Phil's account at any time (Figure 40)

$f(x) = 500 + 40x$

Figure 40. Derek's function for the value of Phil's account at any time.

Following this success, I asked both students to create a function that would work for any initial investment. Derek raised the concern that this new function would depend on two variables: the initial investment and the number of years. After Derek chose 'q' to represent the function, x to represent "the number of years" and n for the initial investment, I began the function by writing q(x,n), and Derek completed the function definition with "=*n*+.08(n)x."

In the remainder of this teaching episode, I questioned the students on the meaning of .08 in the function *q* that Derek created. After the students identified .08 as "the rate of change," I pointed the students in the direction of .08n as being the rate of change. I closed the teaching episode by questioning the students on how to find the account balance for fractions of a year, and we established that a fraction of a year meant that the account earned the same fraction of .08n dollars. The details of these discussions can be found in Tiffany's chapter.

Compound Interest

> The competing Yoi Trust has introduced a modification to Jodan's EZ8, which they call the YR8 account. Like the EZ8 account, the YR8 earns 8% of the initial investment per year. However, four times a year, Yoi Trust recalculates the "initial investment" of the YR8 account to include all the interest that the customer has earned up to that point.

In total, four teaching episodes were devoted to this compound interest task. Due to scheduling difficulties, Derek was unable to attend some teaching episodes at the same time as Tiffany, attending only two of the four teaching episodes while Tiffany attended three. In this particular sequence, Tiffany had two independent teaching episodes (E2T2P2 and E4T3P3) while Derek had only one teaching episode (E3D2) before the students were reunited in E5D3T4P4. Because of this schedule, in particular, because

Tiffany would have two teaching episodes in the same time frame that Derek would have only one, Derek's teaching episode took on a very different character.

In particular, my primary concern in this episode was that Derek would be sufficiently prepared to participate in the discussion I anticipated having in teaching episode 5. With this as my primary concern, my role in E3D2 was more of an instructor that a researcher. I shaped my own behavior toward a goal of having Derek build a function for the compound interest account, rather than towards the goal of building and testing hypotheses about Derek's thinking.

The E3D2 teaching episode followed a similar pattern to my second teaching episode with Tiffany (E2T2P2), but greatly accelerated. I reminded Derek of the simple interest model very briefly and introduced the compound interest within a minute, asking Derek to begin comparing values of Phil's simple interest account and Patricia's compound interest account approximately the two-minute mark.

$$500 + 500 \cdot 0.08 \cdot 0.25$$

$$500 + 500 \cdot 0.08 \, \frac{1}{12}$$

Figure 41. Derek's calculations for the values of Phil's account at a quarter of a year, and at one month. Derek also stated that the value of Patricia's account at these times would be the same.

In Figure 41, Derek calculated the value of Phil's account using the decimal 0.25, rather than the fraction 3/12 as Tiffany had done, indicating that Derek was imagining a quarter of a year as a number of years rather than as a number of months. I next asked Derek to find the value of Patricia's account at the same times, and he very quickly made the observation that Phil and Patricia's account would be identical for any value of time within the first quarter year.

Excerpt 38 -- Episode 3, 00:05:46

1 Carlos: What would Patricia's account be after one month?

2 Derek: Same thing because it doesn't … The eight percent is still coming from the five hundred, not the five ten. Once you get to a quarter of the year-

In this excerpt, Derek describes Patricia's account as being the same as Phil's account until they get to a quarter of a year, indicating that Derek is thinking about all the values that Phil and Patricia's account take on as time varies from zero years to 0.25 years. This description was dramatically different from Tiffany's description of the same accounts in quarter-of-a-year-sized chunks.

Following this, Derek computed the value of Patricia's compound interest account at the end of two quarters and at the end of three quarters (Figure 42). However, the question of the value of Patricia's account at the end of *n* quarters required a little bit more construction. Working from the forms in Figure 42, Derek was able to describe an iterative process for finding the value of Patricia's account at the end of each quarter from zero to *n* quarters.

$500 + 500 \cdot 0.08 \cdot 0.5$

$510 + 510 \cdot 0.08 \cdot 0.25$

$520.2 + 520.2 \cdot 0.08 \cdot 0.25$

Figure 42. Derek's calculations of three account values. Top: Derek's calculation for the value of Phil's simple interest account after half a year. Middle: Derek's

calculation for the value of Patricia's compound interest account after half a year. Bottom: Derek's calculation for Patricia's account after three quarters.

Proceeding further required factoring each of Derek’s forms, just as Tiffany and I did in E2T2P2. Just as in Tiffany’s case, Derek did not remember the distributive property, and had difficulty recognizing the form of his calculations as suitable for factoring. Once those hurdles were overcome, Derek reached the following form for the value of Patricia’s account at the end of one quarter and two quarters (Figure 43).

$500(1 + 0.08 \cdot 0.25)$

$510\,(1 + 0.08 \cdot 0.25)$

$(500\,(1 + 0.08 \cdot 0.25))\,(1 + 0.08 \cdot 0.25)$

Figure 43. Top: Derek's calculation for the value of Patricia's account at the end of one quarter. Middle and Bottom: Derek's calculations for the value of Patricia's account at the end of two quarters.

With a little additional manipulation of these forms, Derek came to abstract from them the general form for the value of Patricia’s account at *n* quarters (Figure 44).

$500\,(1 + 0.08 \cdot 0.25)^{n}$

Figure 44. Derek's calculation for the value of Patricia's account after n quarters.

A brief test, calculating the value of Phil’s account after one month (which is also the value of calculating the value of Patricia’s account after one month), and comparing the result to the one predicted by substituting 1/3 for *n* in the expression in Figure 44 led Derek to the following conclusion.

Excerpt 39 -- Episode 3, 00:29:36

1 Derek: They're both used, for like specific things, so this one [Figure 44] only works with quarters. And then the other one [linear function q

from episode 1] will work with any number but it has to be with the EZ8 account.

In this excerpt, Derek describes the discrete nature of the expression he found in Figure 44: that it only correctly calculates values of Patricia's account on a domain of whole quarters of years. He contrasts it with the continuous nature of the simple interest account function, which works for any value.

Derek's next task was to find the value of Patricia's account after 0.6 years, which he did by finding the value of Patricia's account after 0.5 years, and then using that amount as the basis for a simple interest (linear) form to find interest earned over the final 0.1 year.

$$500\,(1 + 0.08 \cdot 0.25)^{2} + 500\,(1 + 0.08 \cdot 0.25)^{2}\,(0.08 \cdot 0.1)$$

Figure 45. Derek's calculation for the value of Patricia's account after 0.6 years.

Asking Derek to summarize the differences between Phil and Patricia's accounts led Derek to make the following prediction for the graph of Patricia's account value over time, indicating that Derek is imagining Patricia's account as a sequence of linear function pieces, with the steepness of each piece increasing every quarter.

Excerpt 40 -- Episode 3, 00:39:7.492

1 Derek: It'd start… You'd get a line for a quart … Like if you'd have the x axis as number of years, like, for a quarter of a year it's just be straight going up a little bit. Then for another quarter year it'd kinda be a little steeper and keep going up.

In this excerpt, Derek is describing a piecewise linear function, having a constant slope for a quarter of a year before a compounding event. On another occasion, Derek described the same image in his head as “just a bunch of jotty lines growing exponentially.” Also notable here is that Derek explicitly imagines the lines “going up” in between the changes in slope, indicating that Derek is imagining the function being drawn continuously, and taking on value in between the quarters of a year.

The next portion of our conversation concerned developing some terminology for future conversations: ‘quarterly’ as “every fourth of a year;” ‘principal’ as the portion of the investment affected by the 8%, and ‘compounding’ as the act of changing the principal by the accumulated interest.

The final few minutes of the teaching episode were spent beginning to develop a function form for the value of Patricia’s account. Derek’s initial instinct was to treat his calculation in Figure 45 as a function of two variables, rewriting it so that “the two and the point one will change.” Instead of pursuing the idea of the relationship between the “2” and the “.1,” I chose to ask Derek to build the function piece by piece, beginning with the first quarter. This was an instructional decision, partly because piece-by-piece was the method I anticipated using with Tiffany in the subsequent episode, but I also made the suggestion because I did not want Derek developing two independent time indices. Using Phil’s account as a guide, Derek constructed the following form for the value of Patricia’s account over the first quarter:

$$f(x) = 500\,(1 + 0.08x)$$

Figure 46. Derek's function for the value of Patricia's account over the first quarter.

Immediately after typing the above function, and without prompting, Derek restricted the domain, saying that the function would only work up to when x was 0.25.

However, Derek did not know how to represent this restriction as part of the function. I asked Derek to describe to me what values to use, and typed the domain myself.

$$f(x) = 500\ (1 + 0.08x)\ ,\ 0 < x < 0.25$$

Figure 47. Derek's function for the value of Patricia's account on a domain of the first quarter of a year. The lack of a ≤ sign is due to certain software difficulties I had experienced in the past.

E5D3T4P4

This teaching episode opened with the task of describing the work that had been done to date. Derek began, giving the following explanation for the goal of the compound interest task

Excerpt 41 -- Episode 5, 00:00:13

1 Derek: Trying to make a function that would give us whatever the amount that's in her account at any time by using … Since it … it goes by percent each year, but it starts over at every quarter of a year it starts taking the percentage from what's in it then …

Of particular interest is that Derek describes that the goal of the function is to find the amount that's in Patricia's account "at any time," which seems to indicate that Derek is thinking not only of the values of Patricia's account every quarter of a year, but also the values in between. Later Derek appears to be describing a dynamic process that starts over ever quarter, which implies that Derek is imagining the process doing something in between quarters, so that it can start over. Following Derek's recap, I asked Tiffany to explain the function she had developed in his absence. (Figure 48).

$$500\left(1+\frac{.08}{4}\right)^0 + .08(500)x$$

$$500\left(1+\frac{.08}{4}\right)^1 + .08(510)(x-\tfrac{1}{4})$$

$$500\left(1+\frac{.08}{4}\right)^2 + .08(520.2)(x-\tfrac{1}{2})$$

$$500\left(1+\frac{.08}{4}\right)^5 + .08\left(500\left(1+\frac{.08}{4}\right)^5\right)(x-\tfrac{5}{4})$$

$$\frac{5}{4} < x < \frac{6}{4}$$

Figure 48. Tiffany's function pieces for the first quarter (top), second quarter (middle top), third quarter (middle bottom) and sixth quarter with domain (bottom). Domains are omitted from the first three pieces because she had written them previously.

Tiffany initially describes each formula while only discussing the value of the account at quarters of a year, which leads Derek to believe that the function is used only for quarterly values.

Excerpt 42 -- Episode 5, 00:05:00

1 Derek: Like this is for five quarters, and then the other quarter you have you're adding on the eight percent of the last quarter.

Note two things here: that as a result of Tiffany's explanation, Derek understood the functions to be a method for calculating quarterly values, and also, Derek was not thinking about the effect of a quarter of a year on the amount of interest earned. Secondly, Derek referred to "eight percent of the last quarter" instead of "one fourth of eight percent of the last quarter." Neither student was seeing or assigning meaning to the $(x-5/4)$ factor in the function (Figure 48 bottom). When thinking quarter by quarter, this makes sense. There is no need for a variable because there is only one possible time value: a quarter of a year.

The conversation changed abruptly when Tiffany mentioned “excess” time for the first time.

Excerpt 43 -- Episode 5, 00:05:10

1 Tiffany: And then we multiply that by, you're like, with four quarters or however many quarters they give you. But you wanna know the excess so that's why the subtracting of the five 4ths.

These passages (Excerpts 6 and 7) seem to suggest that Derek and Tiffany were regarding $(x - 5/4)$ as an entirely separate part of the equation, which is only now being discussed, rather than an integral part of the second term. Tiffany explains the function now using the example of .6 years that we worked with in the previous teaching episode, splitting .6 into .5 (two quarters) and the “excess” of .1. Following this discussion, Derek and Tiffany give different explanations of the function. The resulting discussion primarily focused on Tiffany, and is described in her chapter.

The second phase of Episode 5 involved predicting and generating graphs of Phil’s simple interest account and Patricia’s compound interest account. The students used their Graphing Calculator to generate a graph of Phil’s account with respect to time. When asked, the students each gave their interpretations of the graph.

Excerpt 44 -- Episode 5, 00:18:01

1 Derek: Like it's growing constantly. But once it gets to one year, it's a total of eight percent higher. And then it grows by still eight percent higher than the five hundred, but just takes that value and its gets up to there each year. Like it's always more money is being

> put in because and keep going like whenever you check it's gonna be even if you're in fraction of a year.

Here Derek is making the claim that for any rational valued number of years, Phil's account will have a different value, which is clear evidence that Derek is imagining continuous (or as he calls it, 'constant') growth. This explanation differs from Tiffany's in a few critical ways, but I did not notice those differences until after the teaching experiment, so the details will be postponed to retrospective analysis.

Next, I asked the students to predict the behavior of Patricia's account. Tiffany was unsure what Patricia's account would look like, and Derek sketched the graph shown below.

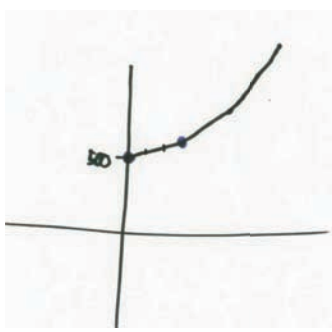


Figure 49. Derek's graph of Patricia's account over time.

I then asked the students to graph the function for the value of Patricia's account with respect to time, Upon completing their graph over a domain of the first three quarters (Figure 50), both students correctly asserted that the lines composing the graph were of increasing steepness, even though there was no visual evidence that the graph was anything other than a line.

Figure 50. Derek and Tiffany's graph of Patricia's account over time.

Constant Per-capita Interest

> The Savings Company (SayCo) also competes with Yoi Trust and Jodan. SayCo's PD8 account policy is as follows: if you have one dollar in your bank account, you earn interest at a rate of 8 cents per year. For each additional dollar, your interest increases by another 8 cents per year. If you have fractions of a dollar in your account, your interest increases by the same fraction, so 50 cents earns interest at 4 cents per year. Here is SayCo's new feature: At any moment you earn interest, SayCo adds it to your account balance; every time your account balance changes, SayCo pays interest on the new balance and calculates a new growth rate. Why is SayCo's PD8 the most popular account?

In this teaching episode, I again focused largely on Tiffany, who struggled to interpret the problem statement. Derek's understanding of the task was much closer to my original intention than hers.

Excerpt 45 -- Episode 6, 00:04:01

1 Derek: What I don't get is like it does it if you have the dollar and then you just put in a dollar at any point does it change right away?

2 Carlos: Umm well it seems like the way this was written that if you put in another dollar it would change right away. But let's imagine for a moment, umm, that we're only putting money in this account one time. We're only investing in this account one time

3 Derek: So does the eight cents interest also affect it? The change of growth rate?

4 Carlos: Umm I'm not sure what you mean by that.

5 Derek: Like if like if you have a dollar you put in a dollar and the growth rate changes right away? Well if you're getting money constantly is the growth rate increasing constantly?

Derek's interpretation of the account was based in a completely different image of the account than Tiffany's. Derek imagined that the account was gaining money at every moment in time, and that as a result, the associated rate described by the task was always changing. This was very close to the interpretation that I intended. Derek was describing an account that was always in flux, and a rate of change based on that account that was also always in flux: an account increasing at an increasing rate. Unfortunately, I could not build on this idea, as it was not compatible with the way that Tiffany was thinking. I postponed this discussion, informing Derek that I would get back to him, and turned to teaching a per-capita perspective of simple interest. This postponement did not seem to have an adverse effect of Derek's confidence in his interpretation. He returned to this image repeatedly throughout the remainder of the teaching experiment.

When I returned to the discussion of "SayCo's new feature," Derek gave a more nuanced interpretation of the task, based on an assumption that the bank worked only in whole cents. He talked about an account for which "every time you get up to one more cent the rate changes." Here Derek saw the rate as changing discretely, but in unequal compounding intervals: whenever one cent is earned: in order for a change to be noticeable (on a statement) would have to change by at least one cent, so one cent is the minimum change. Derek changed his mind when I suggested that the bank would be capable of keeping track of fractions of cents. However, this image of a discrete population growing in continuous time is one that Derek returned to when working with the population biology models. Tiffany interpreted the problem as being compounded annually.

As discussed in Tiffany's chapter, I use the example of two accounts that had been opened at different times but both had a current value of $37 dollars to highlight that the rate of change of the account depended only on the value of the account, and was not dependent on time. From here, I asked the students to construct a function predicting the rate of growth of the account from the value of the account. Derek's function was nearly identical to Tiffany's (Figure 51).

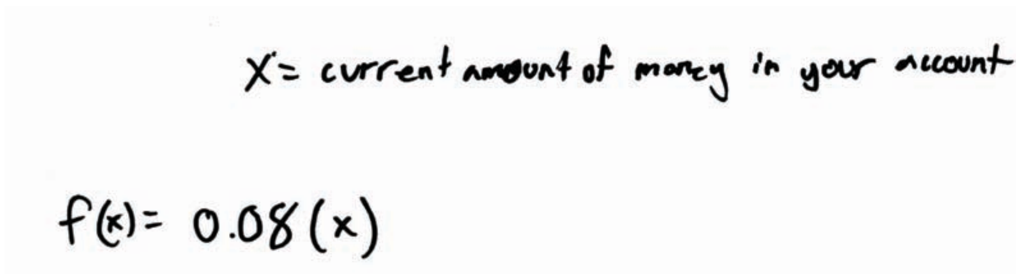


Figure 51. Derek's function for predicting the rate of growth of the account. Tiffany's solution used synonymous wording.

In the debriefing, Pat brought to my attention a problem that I had until that point considered largely solved, which was that there were too many meanings of rate floating around the discussion. The phrase “growth rate” had four different meanings. I viewed growth rate as a number of dollars per year. Tiffany viewed “growth rate” as the amount earned in a unit time. Derek viewed growth rate as “how fast,” and Pat viewed “growth rate” as ambiguous between “interest rate” and dollar per year rate.

Derek’s understanding of rate is best contrasted with Pat’s. Pat distinguished between two rates: an “interest rate” of 8% per year, and a dollars per year rate of .08*N (if *N* is the account balance) and included both meanings in the term “growth rate.” Derek did not make a distinction between growth rate and interest rate. He interpreted “interest rate” as the rate that an account earns interest. In usage, Derek interpreted both terms to refer to both the number .08 and to .08*N dollars per year.

Based on the debriefing, in which Pat pointed out that the students and I had different meanings of “growth rate,” we decided that the focus of the next teaching episode would be on clarifying the meaning of growth rate that I intended.

Compound Interest on the Phase Plane

> Previously: Yoi Trust one-upped Jodan's EZ8 account, which was a simple interest account, with their YR8 account, by adding earned interest to the principal investment at the end of every 3 months (4 times per year).
>
> For the Yoi YR8 account, create a graph that relates the value of the account at any time expressed as a number of dollars and the amount of interest earned during a very short period after that time expressed as a number of dollars per year.

Derek worked on this task for two teaching episodes, E7D5T6P6 and E9D6. The first of the two teaching episodes, E7D5T6P6 was the last time that the two students worked together, before Pat and I decided to take advantage of one of Derek's scheduling conflicts to separate the students for the remainder of the teaching experiment.

Episode 7 (E7D5T6P6) was devoted to interpreting the meaning of the task, specifically, the meaning of the phrase replacing the overly ambiguous word rate: "the amount of interest earned during a very short period after that time expressed as a number of dollars per year." The students' understandings developed very similarly in this episode, and I will only highlight a few points here. The remainder of the details are in Tiffany's chapter.

My intention was to clarify my own usage of rate to the students by describing, essentially, the average rate of change of the function (difference quotient) over a small interval. However, the students were not comfortable with the idea of equivalent rates, an idea made more difficult because ratio never came up at any point in the discussion.

The model of equivalent rates that the students settled on during this discussion was one of partitioning an accumulation. Derek described a model of equivalent rates early on in the teaching episode, and he returned to it with frequency. In the excerpt below, Derek is responding to my question "What's the same for a quarter?"

Excerpt 46 -- Episode 7, 00:7:46

1 Derek: Well not the interest earned, the interest … You'll get the same interest back by the end of the quarter. Like they'll all add up to the same, so-

In this excerpt, Derek described an equivalence of rates based on accumulation: that two rates defined over different time intervals are the same rate if as the same amount of time passes, both rates accumulate (add up to) the same amount of money. Derek's thinking did not change from this position during the course of the teaching episode. However, Tiffany was having difficulty with Derek's extremely brief explanation, and I put the students to the task of filling in the details more carefully.

When Pat introduced the context of a car driving at sixty-five miles per hour, Derek rephrased his partitioning and accumulation model of equivalent rates more clearly.

Excerpt 47 -- Episode 7, 00:30:49

1 Carlos: I'm still going the same speed. I'm still going the same distance in the same amount of time.

2 Derek: It's just uh, if you have a line and for 45 miles and an hour and you want per minute, you just divide it into 60ths. So you have a 60th

> of 45. And then a 60th there's sixty of them so they add up to 45 miles.

In Derek's explanation, one rate was calculated form the other rate by partition, and the rates were equivalent because the same process could be reverse by accumulation: 45 miles per hour can converted to miles per minute by partitioning hours into minutes and the length 45 miles into the same number of partitions, each of length 0.75 miles. The rates are equivalent because the process can be reversed by accumulation: as minutes accumulate to form an hour, the 0.75 mile chunks accumulate to form 45 miles.

I was satisfied with the reasoning at the time, aside from the feeling that an easier solution was eluding me. In retrospect, it appears that Derek's partitioning and accumulation model is a chunky way of thinking about rate. It has more in common with Tiffany's idea of looking at chunk corresponding to a year and cutting it up than it did to Derek's style of continuous reasoning. In fact, Derek never returned to this line of reasoning in any of the following teaching episodes. I attribute Derek's chunky reasoning here to the wording of the task, and the orientation of the conversation towards Tiffany. Specifically, the phrasing I used to explain my meaning of rate "the amount of interest earned during a very short period after that time expressed as a number of dollars per year" built off Tiffany's meaning of rate as a chunk or amount associated with a chunk of time, rather than on Derek's meaning of rate as "how fast."

E9D6

Derek's second teaching episode on compound interest in the phase plane focused on creating the graph, and proceeded very similarly to my work with Tiffany in episode 8. I asked Derek to give interpretations of the YR8 account graph (Figure 20) and of the equations that generated that graph. Like Tiffany, Derek gave chunky explanations, focusing primarily on quarters. This effect was particularly strong when explaining the equations. It is important to note that Derek never derived the equations himself, but relied on Tiffany's work and her explanations of that work. Derek never really demonstrated ownership of these equations. Later, in finding the information that he needed for the third quarter, Derek did not use the equations available to him, but instead chose to re-derive the amounts.

Derek proceeded very rapidly, as he always did. He gave both coordinates of the first point of the trajectory—$500 dollars and 8 percent of 500 dollars per year—as part of his explanation of the task. Derek also had no trouble with the idea of $40 per year for a period of less than a year, explicitly invoking proportionality as part of his explanation

Excerpt 48 -- Episode 9, 00:11:05

1 Carlos: So this rate is forty dollars per year. Umm, does that mean that when a year goes by Patricia has five hundred and forty dollars at the end of the year?

2 Derek: Mmm, no. It's just that's just the rate it's going by, but the rate changes.

3 Carlos: OK.

4 Derek: Probably.

5 Carlos: So what does it mean that it's going, uh, forty dollars per year if it does go for a year?

6 Derek: If it went for the year, it … you would get forty dollars. But if you just go for a quarter you'd end up with a quarter of that forty dollars; so you end up with five hundred and ten dollars at the end of that quarter.

In the above excerpt, Derek explained that a rate of $40 per year does not entail that the account accumulated interest at that rate for a full year, but rather that if the account accumulated interest at a rate of $40 per year for a fraction of a year, the total accumulated interest would be the same fraction of that forty dollars.

When I asked Derek to choose a second point, Derek proposed that the second point he should find should be at one quarter, but rather than this being an example of chunky thinking, Derek already had an image of smooth variation from $500 to $510. He chose one quarter as the time index for his second point not because it was the next chunk, but because one quarter is when the behavior changes.

Excerpt 49 -- Episode 9, 00:17:15

1 Derek: Well as this is growing I guess this sort of just stays the same, because it's … just ‘cause it's growing by that amount for the whole quarter of the year. Then once it gets to the five ten it'll just … jumps up to the forty point because now it's changing by that much at the five ten.

In excerpt 12, Derek explains why he chose one quarter as the time to calculate the next point. He describes imagining the rate remaining at the same value as time progresses for the full quarter of the year, and then calculated the value of the account at the end of the quarter and the new rate. Although Derek says “amount” in this excerpt, he is really talking about the dollars per year rate of change. His choice of the word “amount” was influenced by the wording of the task: “the amount of interest earned during a very short period after that time expressed as a number of dollars per year.”

In by holding the rate constant for a quarter, and then calculating the new value of the account at the end of each quarter, Derek constructed the graph shown in Figure 52. He observed that the line segments were getting longer, and that each discontinuity corresponded to a quarter year of time.

Figure 52. Derek's phase plane trajectory for $500 invested at 8% per year compounded quarterly.

Similar to Tiffany’s episode, I took Derek through a number of other compounding intervals, beginning with five times a year. In exactly the same manner as Tiffany, Derek predicted that the lines would get shorter and the jumps would get shorter. When asked to predict what a phase plane graph would look like if the bank compounded every second, he predicted that the result would be a curve (Figure 53), which I did not expect. When I asked him about his reasoning, he explained that in the step function “the lines are getting bigger and the jumps are getting bigger.” and as a result the limit would be a curve. I interpreted Derek’s words as meaning that he saw the increasing horizontal length of each step and the increasing change in height within each step as synergistic, rather than compensating for each other. Derek drew a curve because he saw everything “getting bigger.” And incorporated that getting bigger into the increasing rate of the curve that he drew.

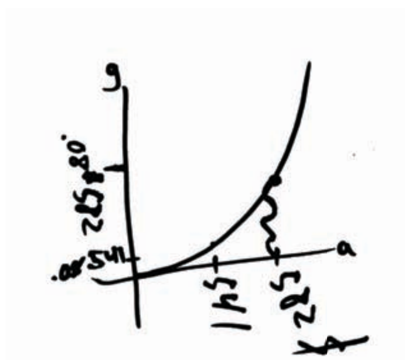


Figure 53. Derek's prediction of the phase plane graph that would result from compounding every second. The numbers on the axes were added later as part of the discussion of why the graph would not be a curve.

Using point by point calculations for the beginning of steps as a guide, Derek defined the function g=.08a as the function of the curve he had drawn in the phase plane. After graphing this function in Graphing Calculator, Derek was very surprised to see that the graph was a line. In fact, Derek believed that Graphing Calculator version was also a curve, just one too shallow to see, even though he has the function definition available to him.

By discussing and making predictions about the graphs of various compounding intervals, Derek came to see the increase in line segment length and the increase in jump height as compensating each other, and that in all cases the points at the beginning of each line segment fell on the function g=0.08a. This argument convinced Derek that that the graph of the limit would be the graph of a line, when an argument from the equation did not, which I take now as further evidence that he predicted a curve initially because he saw the behavior increasing in scale, but did not see the increase in line segment length and increase in jump size as compensating for each other.

Phase Plane Analysis

My two goals for Episode 11 were first to use the work Derek had with compound interest in the phase plane to connect a continuous compounding model with the PD8 constant per-capita growth model from episode 6. The second goal was to have Derek use the phase plane graph to create a graph of the value of the PD8 account over time, essentially, asking Derek to perform an integration. I initially hoped that Derek would do this by making use of rate as a constant rate of change, essentially approximating the function with a frequently compounded account, as I described in the Methods chapter. Derek, however, had his own reasoning method.

This episode began with Derek recapping the entire phase plane series of episodes to date, briefly discussing the goal of the task – to graph the relationship between the value of the account and the rate of growth of the account – as well short recaps of his previous work creating the phase plane graphs for an account compounded quarterly, every fifth of a year, and every second, ending with the phase plane graph of every second resembling a line (Figure 54). I asked Derek what the behavior of the account would be like if the function really were a line:

Excerpt 50 -- Episode 11, 00:05:52

1 Derek: As long as your … the money in your account is growing, then so will the rate of growth will grow. So then it will just keep going up.

The explanation Derek gives here is nearly identical to the explanation that Derek gave of the behavior of the PD8 account in Episode 6 (Excerpt 45). At this point in the teaching episode, Derek's mind was made up. Although I did not ask Derek to create a graph of the account over time until much later, He already had an image in his mind of what the graph will look like (increasing at an increasing rate) and the rest of the teaching episode consisted primarily of me trying without success to pry more details of Derek's reasoning out of him. Without a challenging problem prepared, however, this proved to be quite difficult, although pressing Derek further did result in a few additional details.

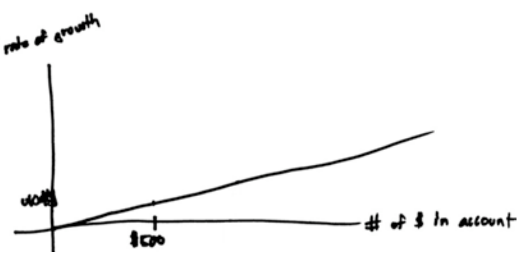


Figure 54. Derek's phase plane graph for a continuously compounded account.

In Excerpt 51 I asked Derek to place a finger on the horizontal axis to represent the value of the account at some time, and to place a finger on the vertical axis to represent the rate of growth of the account at the same time. I then asked Derek to move those fingers while explaining how he was thinking about moving them.

Excerpt 51 -- Episode 11, 00:07:57

1 Carlos: So can you show me how the money in your … in your account is growing, umm.

2 Derek: On that axis?

3 Carlos: By moving your finger along this axis, yeah.

4 Derek: Like starts slow and then just keeps getting faster and faster.

5 Carlos: OK umm and what about the rate of growth?

6 Derek: It would also start slow and keep getting faster and faster.

In Excerpt 51, Derek is engaged in a very complex reasoning process that he explains in very few words. He imagines that as time passes, the value of his account will increase, and that as the value of the account increases, the relationship between account value and rate causes the rate of change of the account value to increase. Simultaneously, Derek is also imagining that as the rate of change of the account value is increasing, the account is growing "faster and faster" and that as a result, the rate of change, tied proportionally to the account value is also growing "faster and faster." Derek elaborates on the role of proportionality in a later excerpt.

Excerpt 52 -- Episode 11, 00:21:11

1 Carlos: What would this graph tell you about that bank policy?

2 Derek: It would tell you that whatever amount of money in your account or whatever you have, it'll grow by eight percent of that amount dollars per year. And what … whenever it grows then it'll change the rate to a faster growth.

Excerpt 52 shows that rate proportional to amount is a key idea in Derek's reasoning about the behavior of this graph. Because the rate is always .08 times the account value, every increase in account value is simultaneously an increase in account rate of growth, which is what leads to the conclusion that both the rate and the account are growing faster and faster (as seen in Excerpt 51). In retrospect, Derek's statement that "it'll grow by eight percent of that amount of dollars *per year*" is significant. It suggests that he was keeping separate two ways of thinking about "growth" that lurked within the discussion: (1) the rate of change of the value of the account as a function of time, as measured in dollars per year, and (2) the rate of change of the rate of change of the value of the account in relation to the value of the account (the constant rate that appears in the phase plane graph). Symbolically, he had separated the idea of $\frac{dV}{dt} = .08V$, where $V = f(t)$, V measured in dollars and t measured in years, and the idea that $\frac{dr}{dV} = 0.08$, where $r = \frac{dV}{dt}$, r measured in dollars per year, V measured in dollars, and 0.08 being the constant rate of change by which r changes with respect to V, measured in (dollars per year) per dollar.

Derek identified the graph of the line in the phase plane as being a phase plane graph of the PD8 account (Episode 6) that had a constant per-capita rate of 8 cents per dollar per year. When I asked Derek to graph the account PD8 account over the first two seconds, Derek created the graph shown in Figure 55 bottom.

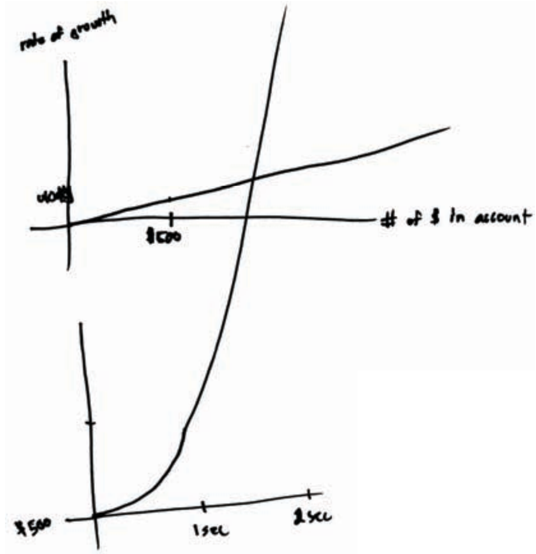


Figure 55. Derek's phase plane graph for a continuously compounded account (top).

Derek’s graph of the account value over time (bottom).

A thread that came up multiple times during the discussion was the issue of whether or not this was a reasonable policy for a bank to use. This discussion essentially took two perspectives. From the compounding continuously perspective, the account was impossible, because every interval required an infinite number of calculations to predict. Each account value determined a rate, and reach rate determined the next account value, except that in a continuous model, there never is a next account value. This was not the perspective that Derek took.

Excerpt 53 -- Episode 11, 00:25:04

1	Pat:	Does this sound circular to you?
2	Derek:	Like what do you mean?

3 Pat: [cough] Well if they're gonna add money to your account constantly, and, but every time they add money to your account, they're going to re-calculate interest and add that to your account.

4 Derek: Mhm [laugh]

5 Pat: So does that sound circular?

6 Derek: Yeah.

7 Pat: What sounds circular about it?

8 Derek: Cause it just keeps like going over and over on what's going on all the time.

9 Pat: Does it sound impossible?

10 Derek: Little bit. [laugh]. Once you get down to like the very, very short amount of time. But I guess it matters like if it's better than going every quarter of a year, because it's growing it's getting faster in between then, too.

11 Carlos: Ok, so what do you mean by when you get down to the very, very little amounts of time it starts to become impossible?

12 Derek: Because it's not noticeable like, like how this is growing always, like. [laugh]

When pressed to decide if continuous compounding was impossible, Derek only committed as far as “a little bit.” and his explanation for what was impossible about it was not that the function could not be found, but rather that it would be impossible to distinguish compounding continuously from a sufficiently small compounding interval.

The second perspective is the perspective from the point of view of the differential equation: that at every moment the account as an amount and a rate of growth and the rate is proportional to the amount. This perspective does not detail how to find a function (without calculus), it simply specifies a function, and whatever function satisfies those properties is the one the bank should use. Excerpt 54, below, shows that Derek had the second perspective—rate as a function of account value. At the end of the interview, Derek was very explicit that finding such a function for the account would be possible, although he didn't know how to do it.

Excerpt 54 -- Episode 11, 00:52:21

1 Carlos: Does this policy that we most recently talked about … what … Would that be a reasonable way for a bank to work?

2 Derek: I guess so.

3 Carlos: Well so far we've had to you know, calculate over seconds, and tenths of seconds, and smaller than microseconds. Could a bank really do that?

4 Derek: Well if they have now you make the function so it grows like that. But once you have that [function], you can just go up to whatever's in your account find out how fast it's growing.

5 Carlos: So if you had the function like all sorted out ahead of time then you could use this policy.

6 Derek: Yes.

7 Pat: I- is this like any function that you've ever met before?

8 Derek: No. [laugh].

The circularity of phase plane analysis clearly did not bother Derek because he made use of this circularity in describing the behavior of such an account with his fingers (Excerpt 54). By making use of this circularity to imagine the behavior of the account in time, the graph of the function predicting the account value with respect to time, (Figure 55). Derek was also open to the idea that there was a function that the bank could use that described the behavior of the account in time, despite the fact that Derek could not imagine finding an equation that would result in such behavior. Derek was open to the idea of a function that he did not fully understand being the solution to a problem, which Rasmussen identified as a key perspective in understanding differential equations (Rasmussen, 2001).

The Malthus Model

The originally designed task for this episode was to ask the students to evaluate the good and bad points of the Malthus model. In this teaching episode, I took a much more informal approach than I originally planned, asking Derek about members of his own family, the "Kellies." I chose this approach for two reasons: firstly because it made the model more personal to Derek, and secondly because asking about a subset of the population allowed me to introduce a rule that made the Malthus model more realistic: say that a person is only considered a "Kelly" if they are the son of a Kelly. This rule removed much of the complexities of human biology and society from the population that we were discussing, such as interactions between sexes. Using this simplification, I imagined leading Derek through an individual perspective approach of deriving the model resulting in a per-capita rate of change that could be compared to the PD8 account model.

I opened this teaching episode by estimating that there are 25,000 Kellies in the world. I asked Derek to list ways in which the population of Kellies could gain new members or lose members. With a little bit of prompting, Derek produced the list shown in Figure 56

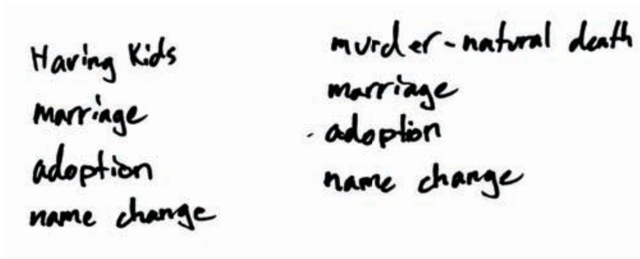


Figure 56. Derek's ways to gain (left) or lose (right) Kellies.

I then proposed my simplification that we only consider a person a Kelly if they are the son of a Kelly. Derek verbally crossed off “marriage, adoption, and name change” from his list of ways to gain or lose a Kelly. I asked Derek what advice he would give to his father to insure that the population of Kellies grows, and Derek responded “have more kids.” This led to a discussion of replacement and an informal introduction of the basic reproductive number.

Excerpt 55 -- Episode 13, 00:07:55

1 Carlos: Umm how many [Kellies] would he have to have to make sure that that the population grows? What's the minimum he could have?

2 Derek: Two. ‘Cause if you have just one you're really not increasing. ‘Cause it's like so when my dad dies and I'm the only son then you just gained one lost one so.

Derek's idea of "gain one lose one" played a key role in later model refinements, because the framework of each Kelly being replaced enabled a later model simplification of imagining that rather than Kellies being replaced, that Kellies were immortal. Derek's "gain one lose one" idea was also necessary to developing the idea of per-capita rate of change in the population. In counting descendants, 1 descendant represents no growth because the descendant replaces the original. This differs from a per-capita rate of change where 0 represents no growth. By focusing the discussion on "spare Kellies" rather than on number of descendants, I was able to assist Derek in creating a rate of individual contribution to the population, where 0 represented no spare Kellies

I with proposed that "Every [Kelly] produces a spare [Kelly] every 25 years or so." When I asked Derek if that was a rate, Derek rephrased my estimate as "You get one [Kelly] per [Kelly] every 25 years." I asked Derek if the rate could be phrases as "one twenty fifth of a [Kelly] per year per [Kelly]." And he stated that there could not be a fraction of a person. In Derek's understanding of this per-capita rate, proportionality of number of Kellies created by an individual to time did not exist.

Derek was initially reluctant to imagine time periods below a twenty-five year interval, but unlike Tiffany, this is not because he was counting in chunks, but rather because Derek was restricting the domain in order to restrict the range (line 2). When I asked Derek about the population rate, He readily accepted that a population of 25,000 [Kellies] would have a rate of 1000 Kellies per year.

Excerpt 56 -- Episode 13, 00:16:21

1 Derek: ‘Cause if you have like one twenty fifth of a [Kelly] per [Kelly] there's not one [Kelly] there's twenty five thousand. So like if you like if you had twenty five [Kellies] you'd get one twenty fifth of a [Kelly] per [Kelly], You have twenty five [Kellies]. You get one [Kelly] a year.

Derek’s proposal that 25 Kellies produce 1 Kelly per year at a rate of one twenty fifth Kellies per year per Kelly even though the same situation is impossible to Derek at the individual level raises the question of how Derek imagines the relationship between individual behavior and the population. There are a number of possibilities that I formed after the teaching experiment but never asked Derek about. One possibility is that Derek imagines the Kellies combining effort to produce a new Kelly, as in the classic joke: “If one man can dig a post hole is sixty seconds, then sixty men can dig the same post hole in one second.” The second possibility is that Derek is imagining that the Kellies reproductive cycles are evenly staggered.

Derek recognized the per-capita rate as the same type used in the PD8 model. After telling a story about Thomas Malthus proposing to use the PD8 financial model for population growth, I asked Derek to evaluate the good and bad points of such a model. Derek’s primary concern was that the model did not account for disasters. With a little bit of prompting, asking Derek to imagine large and small number situations, Derek also concluded that the model was inaccurate if the population was too large (did not account for starvation and wars), or if the population was too small (fractional people became too significant).

The small population situation was particularly interesting, because during this conversation, Derek repeatedly tried to make sense of a nonsensical model, describing the growth as a step function.

Excerpt 57 -- Episode 13, 00:29:47

1	Derek:	You can't progressively get up to one [Kelly]. It just hasta suddenly have one [Kelly].

Derek imagined the population growing as a step function, with no growth until the population could be incremented and the rate compounded. Rate in this model then was not a rate of linear growth or continuous variation, but a way of keeping track of the time until the next step. However Derek also imagined the model as describing continuous growth.

Excerpt 58 -- Episode 13, 00:38:45

1	Carlos:	OK. So can you explain to me what this population … what this model says – the … the model that we've created – about how the population of [Kellies] would grow.
2	Derek:	It says that they grow by like … You have one [Kelly] and then another [Kelly] slowly appears like from bottom up. Starts at the feet and just grows, and then all of a sudden it's all there. And then you get more and more and then more just start popping up.

In this excerpt, Derek is describing a continuously changing population, one where each Kelly "slowly appears like from bottom up." Throughout the remainder of the teaching experiment, Derek repeatedly jumped between the continuous form of the model and his more realistic step function version, and it was not always clear where the boundary between these models was for Derek. Much of this confusion is attributable to lack of clarity on my part. I didn’t always specify which model I wanted Derek to use.

Derek’s two different ways of thinking about the model are reflected in the two graphs that Derek drew to show the population over time. Derek’s first graph is once again based on an ambiguous phrasing on my part.

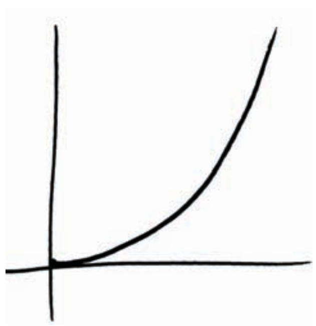


Figure 57. Derek's first graph of what the population of Kellies looks like over time.

Excerpt 59 -- Episode 13, 00:49:38

1 Carlos: OK. So let's sketch out a graph of what the population of [Kellies] looks like over time.

2 Derek: [Draws Figure 57]

3 Carlos: OK. Now is this population of [Kellies] really a smooth curve?

4 Derek: In the fantasy world.

When Derek refers to “the fantasy world” he’s talking about the world in which a Kelly appears smoothly “from the bottom up.” When I asked what the population would look like “in real life,” Derek sketched the step function shown in Figure 58. Derek described his intentions for the graph as each line segment is getting shorter, and each discontinuity jumps up by one Kelly.

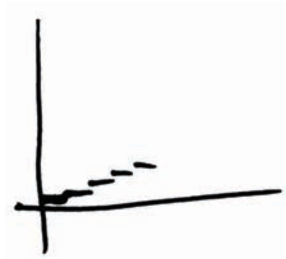


Figure 58. Derek's second graph of what the population of Kellies looks like over time "in real life," and if the ‘curve’ were viewed very zoomed in.

This teaching episode marked the last teaching episode in the “official” teaching experiment. However, Derek graciously volunteered to participate in two additional follow up interviews two weeks later, during which we completed the teaching experiment with a discussion of the Verhulst model. These follow-up interviews are numbered episodes E14D9 and E15D10.

The Food Model

Since it had been two weeks since Derek’s last teaching episode, I opened this teaching episode with an extensive review of the previous episode’s population modeling discussion, beginning with a review of the situation, and then into a discussion of per-capita rate and the (population) growth rate, and finally a discussion of the graphs. Derek again showed signs of thinking of population discretely. Throughout the episode, Derek switched between two compounding models for population: compounding continuously, or compounding every time the population changes by 1.

As a precursor to the Verhulst model, I focused on the high population situation of 70 billion Kellies, which we set as our carrying capacity. Derek described this scenario from a death perspective rather than from a birth perspective, which caused us both a little bit of difficulty, because my design was based around a birth perspective with no deaths.

Excerpt 60 -- Episode 14, 00:13:01

1 Derek: But once you have, so, a whole bunch with them dying off, the rate drops to zero. It just stops growing.

I still have difficulty understanding the intent of Derek's description here in Excerpt 60. It appears from the description "they start dying off" that Derek is imagining the population decreasing. However, Derek also says that they "stop growing" and that the rate "drops to zero" indicating a steady population. One possibility is that Derek is imagining an equilibrium where births are balancing deaths. However, Derek never said anything to that effect. He nearly always focused on deaths to the exclusion of births.

In order to address these issues, I introduced two modifications to the model. First I leveraged our episode 13 discussion about replacement to introduce the idea of immortal Kellies. Secondly, I stole a page from Edelstein-Keshet (1988), and introduced the idea of food being needed to reproduce. Based on the idea of food availability affecting reproduction, Derek imagined a per-capita rate slowing as the population increased. During this discussion, I asked Derek to imagine that population could be used as a rough guide to how much food was available to each Kelly, however we never defined a functional relationship between population and food availability as Edelstein-Keshet did.

Excerpt 61 -- Episode 14, 00:21:33

1 Derek: Since it's a small number they each have a large amount. But once you get to the seventy billion, you're spreading that same amount of food. But instead of tw … like twenty five thousand you're spreading it to the seventy billion. So everyone gets a smaller portion. So then they wouldn't produce kids as much.

In this excerpt, Derek describes the relationship between three quantities: the population, the food that each person gets, which decreases as population increases, and the per-capita rate at which individuals produce kids, which decreases as food decreases. Based on these relationships, Derek created a graph of the per-capita rate of change of the population as a function of population. For comparison, I also asked Derek to make the same graph for the Malthus model (Figure 59).

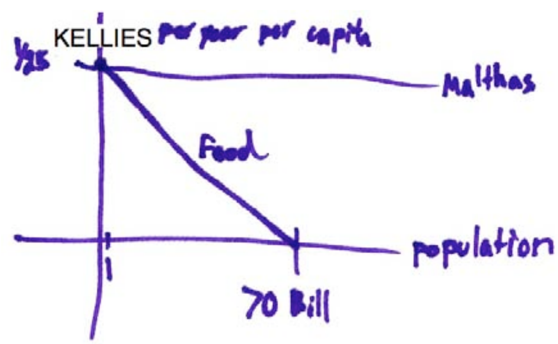


Figure 59. Derek's graphs of the per capita-rate of the population as a function of population.

I then questioned Derek the implications of this food graph (Figure 59) for the rate of change of the population as a whole.

Excerpt 62 -- Episode 14, 00:29:39.

1 Carlos: So when there are two [Kellies], how fast is the population growing?

2 Derek: Twice as … It's slower than before.

3 Carlos: So.

4 Derek: Well not slower, but below double.

In this excerpt, Derek demonstrates that he was distinguishing the ideas of population rate of change (which increases from a population of one to a population of two) from per-capita rate of change, which decreases as population increases.

I asked Derek to write an equation for the food graph, which Derek wrote using the slope intercept form of the line. While writing this equation, Derek did not attend to the meaning of the slope as a rate of growth. This resulted in a brief moment of confusion when Derek used a positive slope rather than a negative slope. This sign error was quickly resolved by graphing his equation in Graphing Calculator.

$$y=\frac{-1/25}{70\,bil}x+1/25$$

Figure 60. Derek's function for the food model. y is "the rate of growth for each [Kelly]." x is the population.

I asked Derek to calculate the rate of an individual in a population of 10, and from there he concluded that the rate of the population would be 10 times the rate of the individual, reasoning that each Kelly had a rate, and that the rate of the population was the sum of 10 Kellies that all had the same individual rate. Based on this reasoning, Derek wrote a function for the rate of change of the population as a function of population (Figure 61).

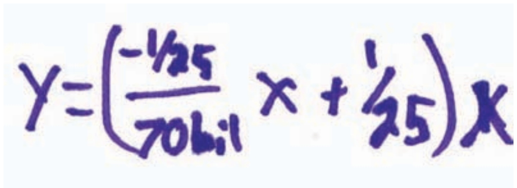


Figure 61. Derek's differential equation for the food model. 'y' is the population growth rate in Kellies per year, x is the population.

Finally, I asked Derek to anticipate what the graph of his population growth rate function (Figure 61) would look like. Derek graphed something unexpected. Despite our earlier discussion of the population rate increasing from one Kelly to two Kellies, Derek graphed a population rate that always decreased with population (Figure 62). I was unable to question him on it before time ran out for the interview.

Figure 62. Derek's initial phase plane graph of the food (Verhulst) model.

Logistic growth

I opened Derek's final teaching episode with a recap of the food model. This time Derek gave a different description than he had during the previous teaching episode four days previously.

Excerpt 63 -- Episode 15, 00:02:26

1 Derek: So like if you're … the … growing at a certain rate, because if you have that much food your food's cut in half, so your rate is going to be half as fast. And then … So for each one then when you put 'em together … Oh, I think I get it. So if they're going at one tenth of a [Kelly] per whatever, and it goes down to one fifth and, so, er, no one twentieth.

In this excerpt, Derek described a model where twice the population meant that each individual grew at half the rate (as a consequence, the growth rate of the population as a whole would be constant). Derek was aware that this was different than the model we had worked with previously, but said that he forgot the original model.

I suggested to Derek that there was a difference between the food available to an individual and the food that that individual consumed, asking him to imagine the situation in which he and his brother were the only two people left in the world. Derek agreed that the food he consumed would not change very much compared to when he was alone in the world, and that his person rate of growth (per-capita rate) would not change very much as a result.

I next asked Derek to describe the graphs that he had constructed during the previous two teaching episodes. We began with the graph of the per-capita rate as a function of population shown in Figure 59. Derek explained the Malthus graph as saying that the rate of growth "for a single [Kelly]" stays the same, and that a thousand Kellies would grow one thousand times as fast as one Kelly. When I asked Derek about the food model, he described a situation in which food decreased with population, and the each Kelly produced new Kellies slower as the population increased.

I last asked Derek to explain the graph he had drawn at the end of the last teaching episode (Figure 62). In episode 14, I had asked Derek to graph the food model in the phase plane, but Derek of episode 15 interpreted the graph as a graph of the per-capita growth as a function of population.

Excerpt 64 -- Episode 15, 00:15:15

1	Carlos:	If I asked you to label the axes of that graph how would you label them.
2	Derek:	The same as these probably.
3	Carlos:	OK, so population and [Kellies] per year per capita. And [Kellies] per year per capita which means?
4	Derek:	How fast a single [Kelly] produces [Kellies]
5	Carlos:	OK, umm, would it surprise you if I told you that you drew that graph when I asked you to graph, umm, the number of [Kellies] and the growth rate in [Kellies] per year for everybody?
6	Derek:	No, ‘cause now I remember. [laugh].
7	Carlos:	OK
8	Derek:	But it I guess it also works for single [Kellies].
9	Carlos:	So, it works for single [Kellies] and it works for everybody?
10	Derek:	Yes. Yeah.

11 Carlos: So how would I interpret that as working for, er … How would you interpret that as working for everybody?

12 Derek: For everyone you'd have how it's your rate for everyone, and then your population. 'Cause once you get to that. You start at one you're growing at same thing. You start getting more and more and it just starts dropping.

13 Carlos: So what's dropping?

14 Derek: Your rate of growth.

I've left this passage long because a large number of interesting things happen in a very short period of time. Derek initially interprets the graph as showing per-capita rate of change (lines 1-4). Indeed, this could be an alternate graph of the 'food' model, as the only restriction I had placed on the model was that the per-capita rate of growth reach 0 at some carrying capacity. However, Derek is not surprised that the graph is a graph of the phase plane (lines 5&6). Derek goes on to explain that the graph could simultaneously be a per-capita graph and a graph of the phase plane, explaining that it shows how the rate of growth of the population decreases at high population.

It appears in line 12 that Derek is imagining the population growing from one individual to the carrying capacity, when Derek mentions "You start at one," but Derek is not. He's concerned only with the behavior at the end-game: as the population approaches the carrying capacity. This is seen when I query Derek about the behavior of the population at small numbers.

Excerpt 65 -- Episode 15, 00:16:50

1 Derek: One [Kelly] you … you're going … you have … All the food is available to the one [Kelly].

2 Carlos: Mhm.

3 Derek: And then you get two. Half of it's available to you, and then, but you're still eating not-

4 Carlos: You're … you're still eating.

5 Derek: -all of it

6 Carlos: But you're still eating the same amount.

7 Derek: Yeah you're still eating the same amount. And then once you get more and more, the availability … availability goes down so you're not able to eat as much.

In the above excerpt, Derek begins by talking about one Kelly and two Kellies, but quickly transitions to talking about large populations “once you get more and more.” It was only when I pinned Derek down to talking only about the one and two individual situations and finding the growth rate for each that Derek corrected his graph of the phase plane (Figure 63).

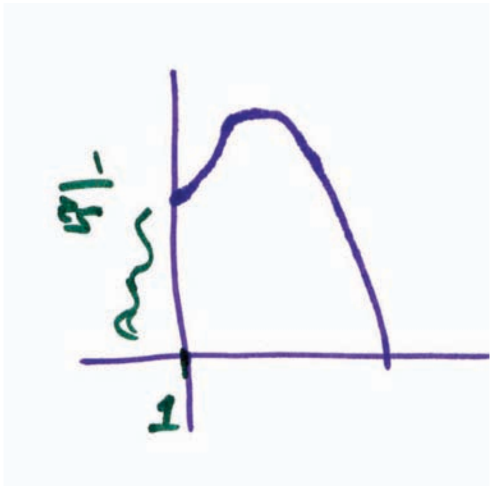


Figure 63. Derek's phase plane graph of the 'food' model. The numbers on the horizontal and vertical axes were written by me after Derek identified them.

Following this discussion was a recap of the meanings of the equations that he had written in the previous episode, and we graphed Derek’s equation for the Phase plane graph of the food model in Graphing Calculator (Figure 64). Derek and I concluded that his graph of the phase plane was generally correct, although the scale was off.

In preparation for asking Derek to graph the population of the ‘food’ model over time, I once again asked Derek to explain what the phase plane of the Malthus model said about the growth of the population over time. Once again, Derek and I had a crossing of meanings here, in that Derek answered about the predicted physical population of people, describing the population as a step function increasing in one Kelly increments, when I had intended to ask about the numerical output of the Malthus model, which increased continuously. This difference of meanings came to a head when I asked Derek to graph the population of the food model over time, and Derek had to stop and ask for clarification, asking me if I meant how the population really grows, or if I meant the growth permitting fractions of Kellies. I specified that I was asking about “the fantasy world.”

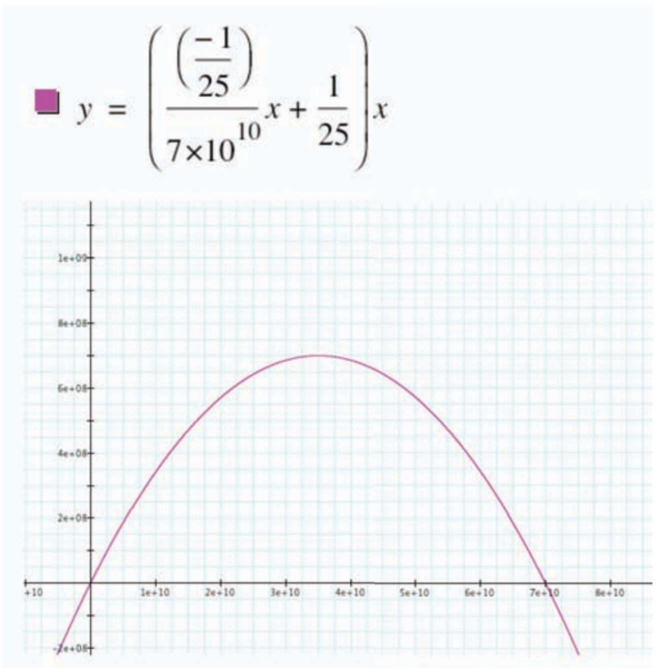


Figure 64. Derek's phase plane equation for the food model and the phase plane graph for the 'food' model rendered by Graphing Calculator.

Derek ultimately gave two explanations of the behavior of the population of Kellies over time. His first explanation (Excerpt 66), used reasoning about the food supplies of the population, and in this case, Derek was relying on his understanding of the situation rather than on the phase plane graph.

Excerpt 66 -- Episode 15, 00:36:59

1 Derek: Alright, so as you keep getting more [Kellies], umm, you lose your number of food, and then at … or you lose your availability, so you go down a little bit, but you're still growing more, because

your creating [Kellies] faster then you are losing food so you're going to be going up until whatever point that

In this excerpt, Derek is again focusing on the end behavior of the population, describing a population that grows slower and slower as the population grows. The phrase "growing more" here is ambiguous with respect to whether Derek is imagining the population increasing at an increasing rate or the population continuing to increase beyond its previous value. The graph Derek draws later (Figure 65) shows the later to be the case. It is possible that Derek is using the phase plane graph as a guide (the video is not clear on where Derek is looking here), but even if that was the case, his description of food shows that he was using a situated image of Kellies and food as an intermediary in this reasoning.

Later, I asked Derek to explain from the phase plane graph, and Derek gave a much less ambiguous answer. In this excerpt, Derek is explaining directly from the graph, pointing and gesturing to it, and the language of his response is very different, referring to specific numbers and explicitly to rate.

Excerpt 67 -- Episode 15, 00:39:19

1 Derek: You have your [Kellies] growth per year, so you can look at from zero from one to one billion. It's grow … it your rate of growth is increasing, so it'll be like going faster across until you get to here, where it'll start slowing down to nothing.

Immediately after Excerpt 67, I asked Derek to sketch the graph. The sketch that Derek created (Figure 65) was consistent with Excerpt 66, but not consistent with Excerpt

67, indicating to me that Derek drew the graph from his imagining of the situation rather than from the interpretation of the phase plane.

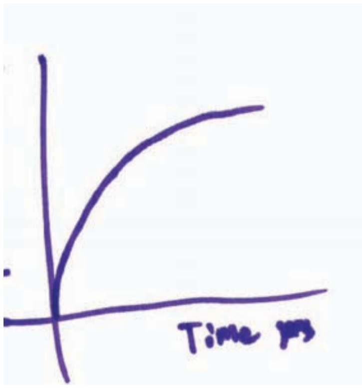


Figure 65. Derek's first graph of the 'food' model population over time.

When I asked Derek to explain his graph he immediately changed it. After drawing his second graph of the population over time (Figure 66), I asked Derek to explain how he could see the time graph in the phase plane graph. Derek explained that the rate of growth was increasing until it reached “3.5 billion” and then decreased. I asked him about the value at the end of the graph that he had drawn, and Derek redrew the graph a final time, adding numbers on the axes reading “35 billion” and “70 billion.”

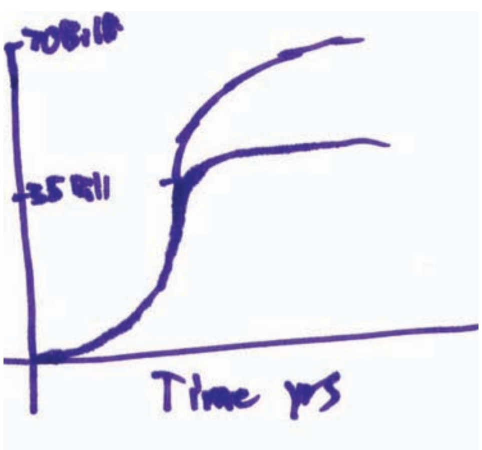


Figure 66. Derek's second and third graphs of the population model over time. The final graph approaches 70 billion.

CHAPTER 6

RETROSPECTIVE ANALYSIS

Following the teaching experiment, I performed a second pass of transcription by converting each transcript file to a subtitle file, meaning that the transcription appeared immediately below a video image as people spoke in it. I then watched the video of each teaching episode with subtitles. This enabled me to both re-acquaint myself with the older teaching episodes from a new perspective of knowing the outcome, and also to catch and correct errors in the first pass transcription.

By watching the subtitled videos I identified certain themes in the students' reasoning that persisted throughout the teaching experiment. The themes explained a great deal about my design of the experiment and about what occurred over the course of the teaching experiment. Of particular interest were the distinct ways in which Tiffany and Derek imagined change occurring. During the teaching experiment, these two ways of thinking about change, which I called "chunky" and "smooth", developed in their thinking into very different understandings of rate of change, per-capita rate of change, and exponential growth.

Chunky and Smooth

A major goal of my retrospective analysis was to better characterize the differences between Tiffany's "chunky" thinking, and Derek's much more continuous variation. My method in this case was to look over the transcripts and select from them excerpts in which the "chunky" or "smooth" nature of each student's thinking was particularly clear. Closer study of these "chunk and "smooth" labeled excerpts revealed a pattern. "Chunky" reasoning occurred when a student imagined a completed change, either a change that had been completed in the past, or a change that the student anticipated being completed in the future. "Smooth" thinking occurred when a student was discussing a change in progress: a process of change that had begun but had not yet completed. This difference is most clearly seen in the following excerpt from episode five. Just prior to this excerpt, the students, working together, had used *Graphing Calculator* to generate the graph of the line $y = 500 + 0.08(500)x$.

Excerpt 68 -- Episode 5, 00:17:17

1	Carlos:	Explain to me what you're seeing. What does this graph show?
2	Tiffany:	This graph shows that his… It starts at five hundred because at zero years it starts at five hundred. And then it increases little by little every year, And not even every year but every like parts of it. Just little by little
3	Carlos:	So every parts, what do you mean by every parts of it?
4	Tiffany:	Like umm even every day of the year. That's a that's a part of a year. So even every day of a year it's still growing [a] tiny tiny bit. You might not be able to see it on that graph cause it's hard to see. Anyway, but if we had divided the year up like we'd done before, We would see that like between one month and then the next month is, it's growing it's like kinda like that.
5	Derek:	Like it's growing constantly, But once it gets to one year it's a total of eight percent higher. And then it grows by still eight percent

higher than the five hundred but just takes that value and its gets up to there each year.

6 Carlos: OK, so what do you mean by it's growing constantly?

7 Derek: Like it's always more money is being put in because… and keep going. Like whenever you check it's gonna be even if you're in fraction of a year.

In line 2 Tiffany describes the graph as growing “little by little,” initially describing that “little by little” as occurring every year. At this point, Tiffany is imagining a series of completed changes: a year has passed, and the account has increased a certain amount, then another year has passed, and the account grows a certain amount, and so on. Shortly afterwards, Tiffany revises her time chunk size to be “like parts of [a year],” But Tiffany is still imagining the parts of a year as occurring in completed chunks. On line 4, Tiffany gives exactly the same explanation of how the graph is changing, but substitutes chunks of “days” for chunks of “years.”

In contrast, Derek’s description of the behavior of the function is much more dynamic. On line 5 , Derek also describes a completed change “Once it gets to one year it's a total of eight percent higher,” indicating chunky thinking, But just prior to this, Derek describes a continuous change in progress: “It's growing constantly.” My asking for clarification reveals that Derek is not using “constantly” here to mean “at a constant rate” but rather “all the time.” On line 7, Derek describes his meaning of “constantly” as “it's always more money is being put in.”

This contrast allows me to define my terms: From this point forward, I say that a student is engaged in “chunky thinking” if that student is imagining completed changes. I say that a student is engaged in “smooth thinking” if that student is imagining changes in progress.

Chunking Units

Tiffany frequently used the particular variety of chunky thinking as defined above, in which Tiffany takes the completed change as a unit, imagining repeated changes of uniform size. For the purposes of distinguishing this type of chunky thinking form other varieties, I will refer to it as “unit-based chunky thinking.” Unit-based chunky thinking can be seen in Excerpt 68 above. On line 2, Tiffany describes the changes occurring “every year” while on line 3, Tiffany describes changes as occurring “every day.” In each case, Tiffany is choosing a completed time change “year or day” as a unit, and imagines subsequent changes as occurring uniformly in that unit. At other times during the teaching experiment, Tiffany used months or seconds as her chunking units.

Although Tiffany shows a marked preference for pre-existing standard units such as years, months, days, or seconds, Tiffany is not limited to solely choosing these sizes as units. An example of this can be seen in the following excerpt, in which Tiffany is explaining the YR8 (compound interest) account policy.

Excerpt 69 -- Episode 2, 00:10:44

1 Tiffany: You would divide up the year into four parts. And then you look at like the next, like one of the quarters of the year and see: ok, after from zero to this quarter of a year, it's [the account balance]

> changed this much. And then you could take that number and do the uh rate thing. And then we'd have to look at the quarter of the year again and see the change there. And then you'd have to redo uh the rate and— it's a little bit more complicated than, um, yesterday.

In this Episode 2 excerpt, Tiffany is imagining the completed change of a "quarter of a year" as her unit, that is, she's imagining repeated completed changes where each completed change occurs in quarter of a year sized chunks, showing that she's not restricted to pre-existing units. There are two points of interest in this example. The first point of interest is that in only addressing the behavior of the account in quarter sized chunks, Tiffany is losing the richness of the behavior that occurs within each chunk, specifically, Tiffany is focusing only on how the change in the account balance changes each quarter, without attending to the details of the linear growth between quarters.

The second point of interest is the manner in which Tiffany created her new "quarter of a year" unit. Tiffany created this new unit by imagining a "year" unit and "dividing it" into "four parts." Indicating that the origin of Tiffany's "quarter of a year" unit is not from imagining a completed change of 0.25 years in isolation. Rather Tiffany imagines a completed change of a full year, and then cuts up that full year of completed change up into one-fourth-year units. Chunk sizes that cannot be reached by dividing evenly do not have meaning to Tiffany. In this next excerpt, from Episode 4, Tiffany is working with the situation in which Patricia invested $500 in an account that's compounded annually. The underline is for vocal emphasis.

Excerpt 70 -- Episode 4, 00:16:14

1 Carlos: Let's say I want to use the same sort of reasoning to figure out the value of Patricia's account after point six years.

2 Tiffany: Point six <u>years</u>? uh kay well we would have to figure out how much of the year that really is. Well six tenths of a year. Like we could figure out how many quarters that is or something and then so the same type of thing like if it's just point six, I don't know right now I don't know how much of a year oops that really is.

3 Carlos: OK, so what would you need to know how mu… to know how much of a year that really is?

4 Tiffany: Umm, we'd need to know how many months – if you wanna look at months – si… point six of a year is.

In this excerpt, Tiffany is expressing two different concerns. The first concern is that point six years initially doesn't have a meaning for her, until she identifies it as "six tenths of a year." One interpretation of this phrase, consistent with Tiffany's chunky thinking and the fraction lessons from her Algebra II class, is that Tiffany imagines "point six of a year" as coming from one year being cut up into ten equal-size pieces, and then taking six of those pieces. The second issue Tiffany raises, also in line 2, is that from six tenths of a year, she doesn't know how many quarters have passed, so she doesn't know how many compounding events have occurred. Tiffany's proposed solution to these problems is to look at the elapsed time in months. Essentially, Tiffany is trying to find a least common denominator for six tenths of a year and a quarter of a year, so that she can count a number of uniform chunks that six tenths of a year is and also know how many quarters have passed, also by counting those chunks, although her proposed chunk size of “month” is not small enough for this to work. This is a result of Tiffany imagining chunks always having the same size. It does not occur to Tiffany to imagine the point six years as some number of quarter of years followed by a remainder of 0.1 years, because those chunks would not be the same size.

Other Forms of Chunky Thinking

Although Tiffany had a strong preference for unit-based chunky thinking, she occasionally imagined completed chunks in varying sizes. One example is coming from her explanation of the piecewise linear compound interest function, in which she described finding a value for “half a year and a month” (Chapter 4, Excerpt 15).

$$500\left(1+\frac{.08}{4}\right)^2 + .08(520.2)\left(x-\frac{1}{2}\right)$$

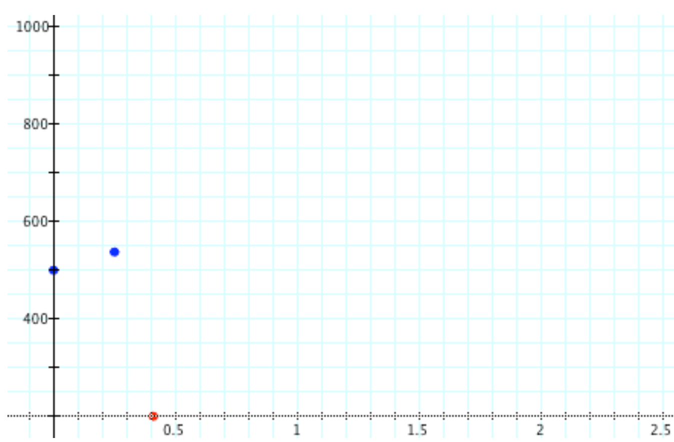
Figure 67. Tiffany's compound interest function for predicting values of the account during the third quarter.

Tiffany used this mixing of chunk sizes only when she could imagine them in separate familiar units. In this case, Tiffany imagined a completed change of one half, and then imagined an additional change of one month. Tiffany could have explained the behavior of this function t in any repeated unit smaller than a quarter of a year, such as months. However, Tiffany also needed to explain the “1/2” as part of her explanation of the function. Since one half is measured in years, but would not serve as a unit-chunk for describing behavior within the function, Tiffany’s explanation mixed two different sizes of chunks: half a year and a month.

Smooth Thinking

Derek's smooth thinking is characterized by imagining change in progress. Specifically, Derek imagined and described situations changing in experiential time. Derek would imagine time passing for the financial accounts (conceptual time) as time passed for himself (experiential time), although on different scales. Because experiential time is inherently continuous, imagining a conceptual time changing in experiential time entails continuous variation.

The importance of experiential time in smooth thinking can be seen in one of the few examples in which Tiffany engaged in smooth thinking. In Episode 5, I showed the students a number of animations of graphs of different account policies and asked the students to explain what they saw. One policy only reported account balances on the quarter of a year, and clients could not access their account balance at any other time. During the discussion of this policy, I showed the students an animation showing a red dot moving along the horizontal (time axis), laying a blue account value dot as it reached each quarter of a year Figure 68.



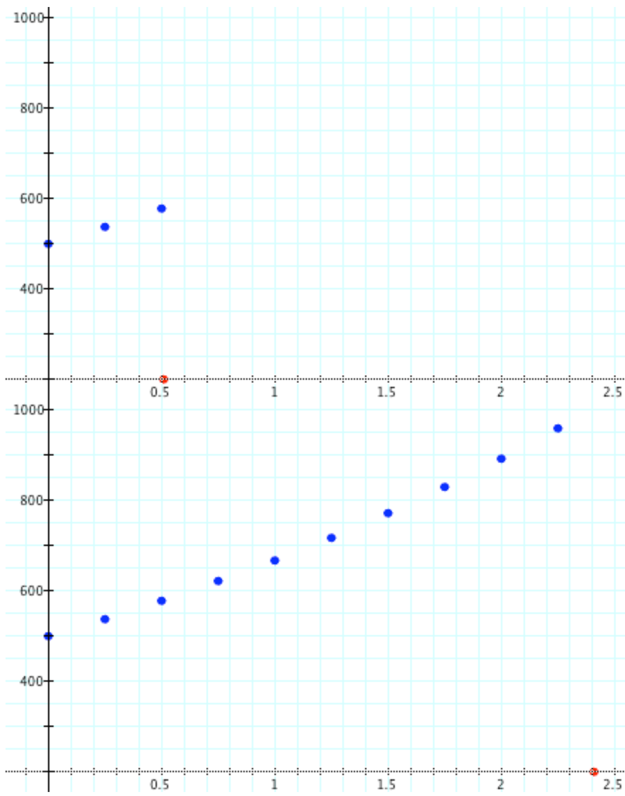


Figure 68. Three frames from the "red dot" animation of an account that only reports values every quarter. In the top frame, the red dot is at 0.4 years, so the animation does not yet show the value of the account at 0.5 years. In the middle frame, the red dot has passed 0.5 years, and the value of the account at 0.5 years has appeared as a blue dot. The bottom frame shows the overall behavior of the animation.

When I asked Tiffany to describe the role of the red dot in the animation, Tiffany showed signs of smooth thinking.

Excerpt 71 -- Episode 5, 00:38:09

1 Tiffany: The [blue] dots are only appearing… You know… It's going, like, red dot is going through all of the points on there… on the

> change… the time line, but the other the points are just appearing at like quarter, half three, fourths.

In the above excerpt, Tiffany talked about the red dot changing in the present tense, and talked about the red dot traveling "through all the points." Both are signs of smooth thinking. Tiffany was able to engage in smooth thinking for two reasons. First, the continuously moving red dot and discretely appearing blue dots created a contrast between smooth and chunky behavior that she needed to talk about in order to explain the animation. Secondly, the animation was playing as she saw it. Tiffany was describing the behavior of the animation while she was experiencing the animation, which placed the movement of the red dot in her own experiential time. These two reasons are related: Tiffany could not have talked about the contrast between moving and non-moving dots had she not experienced a moving dot.

Tiffany showed a very strong preference for thinking in chunks. That asking Tiffany to explain the behavior of an animation of a moving red dot elicited smooth thinking from her shows the importance of an experiential perspective in smooth thinking.

Chunky, Smooth, and Continuity

Smooth thinking, which occurs in continuous experiential time, is most certainly a type of reasoning about continuous variation. However the question is still open as to whether or not continuous variation reasoning can arise within chunky thinking. Tiffany did not show signs of continuous variation within her chunky thinking, however my suspicion is that chunky continuous variation is possible.

In the case of chunky thinking that uses completed changes of variable sizes, the flexibility of the framework should allow for continuous variation so long as the student is aware that the variable chunk sizes they are currently using are not the only possible chunk sizes, and that the chunks can be any real number in size.

In the case of unit-based chunky thinking, it may be possible to imagine continuous variation by coordinating various sizes of chunks, partitioning each chunk into smaller chunks and recognizing that variation on the smaller chunks can be further partitioned without end. This would result in a type of dense variation that approximates continuous variation but does not pass through all real numbers. I have not yet considered the mechanisms by which this approximation of continuous variation might then be used to teach real-valued continuous variation, but I am not willing to exclude the possibility that such mechanisms exist.

Rate

One of the greatest difficulties I had during the teaching experiment was the number of meanings of ‘rate’ that occurred during the discussions. In planning the teaching experiment, I had in my own mind a meaning of rate similar to “slope of a tangent line.” During the Algebra II class in which the students had participated, the students had worked only with linear and polynomial functions, and in the case of polynomial functions, rate was taught primarily as average rate of change, and I did not anticipate that developing an idea of rate from what the students had been taught would be problematic.

8% as a Rate

One of the earliest difficulties I had with rate came from the financial context that the students were working in. Throughout Episode 1, and continuing in some later episodes, Derek and Tiffany identified 8% as 'the rate,' as in the excerpt below.

Excerpt 72 -- Episode 1, 00:16:26

1 Carlos: What does that number really mean? What does the eight percent mean?

2 Tiffany: Umm …

3 Derek: It's the rate of change.

4 Tiffany: Thank you, yeah, that's kind of what I was going for.

This example is particularly interesting because in Episode 1 the students were working only with linear functions. Despite this, neither Derek nor Tiffany used the tools of linear functions taught in class to identify the rate of change of the linear function. They did not identify the m in $y = mx + b$, nor did they calculate a difference quotient. Instead, both students identified the rate from the financial context. In many financial situations, 8%, as an interest rate, is referred to as "the rate", and this is the meaning of rate that the students were using here. Pat also made use of this meaning of 8% when questioning Derek about his meaning of "growth rate" asking the question "Is this an interest rate? Or is it a number of dollars per year that you're talking about?" (Episode 6, 00:5:20).

Rate as Amount

A third meaning of rate was the meaning of rate as an amount earned in a certain period of time. This meaning was used most frequently by Tiffany. The clearest example can be seen in the driving a car scenario that Pat presented in episode 7.

Excerpt 73 -- Episode 7, 00:21:17

1 Pat: If I'm going sixty-five miles per hour what does that mean?

2 Tiffany: That in one hour you've gone, you should have gone sixty-five miles.

In this excerpt, Tiffany described sixty-five miles an hour in terms of chunks. She imagined a completed change of one hour, and imagined a completed change of sixty-five miles. This chunky thinking initially threw Tiffany off her stride when Pat asked a follow-up question

Excerpt 74 -- Episode 7, 00:24:22

1 Pat: Can I travel for just one second at sixty-five miles per hour?

2 Tiffany: No. You have to do… You would have to do, um… Well, yeah, you could.

When Pat asked Tiffany if a car can travel at sixty-five miles an hour for just one second, Tiffany's initial reaction was "no," because she was imagining traveling for one hour. Her quick correction of herself came from yet another meaning of rate which I will discuss a little bit later.

So far, the evidence has shown that Tiffany was thinking of rate as occurring in discrete chunks, but I have not yet shown that Tiffany was thinking of 'rate' as an amount. I offer two arguments in favor of a 'rate as amount' interpretation. The first is that in Tiffany's reasoning of unit-based chunky thinking, the unit of time that regulates the size of the chunks can be ignored, because the chunks are always the same size, reducing a coordination of a chunk of amount with a chunk of time to simply a chunk of amount. For a second point, I turn your attention to Tiffany's use of the word 'rate' in a financial context, as seen in this excerpt from Episode 2, in which Tiffany and I were discussing the simple interest model.

Excerpt 75 -- Episode 2, 00:01:38

1 Carlos: If Fred wants to, umm, check his account every month, ummm… How the bank would figure out how much interest he earned in that month.

2 Tiffany: Mhmm

3 Carlos: So, can you tell me a little bit about that?

4 Tiffany: Well, I think we decided that to check maybe every month, or every parts of months, we could like take the x and divide it into months or days or… and then that would divide up the rate too. So we could like find out how much let's say after a day there's only been like this much change. So kinda that kinda thing.

In the above excerpt, Tiffany described a process of finding the amount of interest earned in a month by taking the rate of a year, and dividing the year up into month sized chunks, and then also dividing the "rate" up into an equal number of chunks. From the point of view of someone who distinguishes rate from amount, it may appear that by dividing the rate, Tiffany was converting from a rate of dollars per year to a rate of dollars per month, and this would be correct. However, the point I wish to make with this excerpt is that in thinking this way, Tiffany never needed to make a distinction between rate and amount. If one thinks of the rate as the amount earned in some unit time, dividing up the rate is the same as dividing up the amount, and in this way, Tiffany found interest earned in a month.

Rate as a Chunky Proportion

Excerpt 75 above, while it is an example of rate is amount, is also an example of a more sophisticated meaning of rate than the one Tiffany describes in the speed example (Excerpts 73 and 74). In the case of the speed example, Tiffany described a rate that is simply composed of two solid chunks: sixty-five miles an hour means traveling for one hour and traveling for sixty-five miles. However, in excerpt 74 when Tiffany changed her mind to say that someone could travel for sixty-five miles per hour for one second, Tiffany was using a different meaning of rate. This fourth meaning of rate is an extension of rate as amount, and a clear example of it can be seen in excerpt 75 when Tiffany proposes dividing both the time and the rate by the same value.

Rate as a chunky proportion describes a relationship between two pairs of completed changes. In order to talk about this more readily, I'll use the financial context from episode 6. In the case of rate as amount, Tiffany imagined a rate of forty dollars per year to mean a chunk of forty dollars earned in a chunk of a year. However, this meaning of rate broke down when Tiffany was asked to calculate values of the account over fractions of a year, as in the case of an account compounded quarterly. In these situations, Tiffany created a new rate by cutting up the old rate into smaller chunks. In Excerpt 8, Tiffany wanted to convert the number of dollars earned in a year to the number of dollars earned in a month, so she divides the chunk of time and the amount earned in that time into twelve parts. This way of thinking is 'chunky' because it relies entirely on imagining completed changes: imagining a change of a year and the corresponding change in account value, and then imagining a change of a month and the corresponding change in account value.

Although Tiffany used the calculations to find equivalent rates (meaning ratios), Tiffany did the calculations without a sense that a ratio had been preserved. Tiffany understood that change of forty dollars in a year in someway could be extrapolated to a change of a twelfth of forty dollars in a twelfth of a year, and vice versa, but she didn't think of forty dollars per year as meaning the same thing as three and a third dollars per month. This is because Tiffany was coordinating chunks (a forty dollar chunk and a one year chunk) without making a multiplicative comparison across the chunks (the number of dollars is forty times as large as the number of years).

In the following excerpt, Tiffany and Derek were working with the compound interest account, in which the rate (ratio of change in dollars to change in years) itself changes every quarter. However Derek has said that within a quarter the rate is the same. I posed the following question to the students.

Excerpt 76 -- Episode 7, 00:11:00

1 Carlos: So, let's think about it this way. Umm. Over that quarter of a year I earn a certain amount of interest and a certain amount of time has

passed, and the time has passed is a quarter of a year. Now if I look at, umm, say, oh, even a hour in that quarter of a year, I earn a certain amount of interest, and a certain amount of time has passed. An hour has passed. Now, I understand… I understood [Derek] as saying that in some way even though they earn a different amount of money and a different amount of time has passed, those are the same in some way. So, how are they the same?

2 Tiffany: Oh.

3 Derek: They really … they depend on each other for them to exist. Like you need in order to get your, um, how much money you'd earned at the end of the quarter, you'd know like that it's a quarter, like.

4 Tiffany: Umm, they both use the same, umm, bank… what the bank gives you to use, like, eight percent… or the, umm, point oh eight. They both use that. And I'm just looking at little similarities 'cause that'll help with the big picture.

5 Carlos: Well, I mean that's true of other quarters, too.

6 Tiffany: Yeah.

7 Carlos: Like other quarters we still use the eight percent, but something about them is different.

In this excerpt, I address precisely the issue of what is preserved across Tiffany's Chunky proportion. Of the three of us in this discussion, none of us were able to give an adequate answer to this question. At the time, I was thinking in terms of chunky proportions as well, rather than in ratios. The best answer came from Derek, who proposed, essentially, that an accumulation of these hour-sized chunks would result in the quarter-sized chunk, and that the same was true for the corresponding dollar amounts.

None of us made a multiplicative comparison of change in dollars to change in hours, in part because of the phrasing of the question itself. By converting from years as the unit to hours as the unit, I obscured the ratio in question. Tiffany, who creates a new unit every time she converts, is placed in the same dilemma, in comparing a number of dollars earned in a year to a number of dollars earned in a quarter of a year, Tiffany does not see a preserved ratio because she treats “a quarter of a year” as a unit. Both ratios are amounts of dollars over one unit, so even if it occurred to Tiffany to compare ratios, the ratios would be different.

Rate as “How Fast.”

Calculating a rate of change using the tools the students had learned in class required chunky thinking. Average rate of change as difference quotient depends on chunky thinking in that it imagines a completed change in y and a completed change in x. Derek’s smooth thinking was not compatible with thinking about rate as the result of calculational operations. Derek imagined rate more as an index labeling “how fast” some quantity was changing rather than a measurement. A larger rate number meant that the quantity was increasing faster. This very informal way of thinking served Derek well in phase plane analysis. When examining the linear graph in the phase plane (Chapter 5 Excerpt 51), Derek’s quick recognition that the account would be increasing “faster and faster” came from his equating of a higher rate with “faster.”

Thinking about rate as an index of "how fast" would also explain the difficulties Derek had distinguishing between .08 (or 8%) as the financial interest rate and .08*n* as the dollar per year rate of change during the simple and compound interest models early on the teaching experiment. Derek saw both numbers as indices of how fast the account grows. Increasing from a dollar per year rate of 40 to 40.8 would mean that the account would be growing faster, but increasing the policy from 8% to 9% would also mean that the account was growing faster. These difficulties nearly entirely disappeared once Derek and I began studying per-capita rate of change in more depth. Having per-capita rate of change as a way of thinking enabled Derek to distinguish between how fast the account was changing (in dollars per year) and how fast the rate of change of the account was changing (in dollars per dollar per year, or dollars per year per dollar).

Waiting Time

Another meaning of rate of change that Derek used was rate of change was a number that specified waiting time. He used this meaning of rate of change in situations where he imagined that the population could only take on discrete values. Derek used this idea briefly in the PD8 model when he imagined that the account could only take on whole cent values, but the idea blossomed in the population biology model.

Excerpt 77 -- Episode 13, 00:28:21

1 Carlos: one twenty fifth of a [Kelly] per year

2 Derek: which doesn't make sense so you'd have to expand the years to twenty five years you get one [Kelly]

In Excerpt 77, Derek described a process by which one Kelly waits 25 years and then suddenly makes a Kelly all at once, resulting in a discontinuous population which he graphed as a step function (Figure 69).

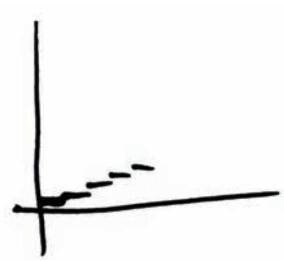


Figure 69. Derek's second graph of what the population of Kellies looks like over time "in real life," and if the ‘curve’ were to be viewed very zoomed in.

In this step function, Derek described each discontinuity as jumping up by one Kelly, and each segment as getting shorter, showing that Derek imagined the waiting time for the population to make the next Kelly as decreasing as the population and the rate increased.

Rate as a Smooth Proportion

One meaning of ‘rate’ that never occurred during the teaching experiment, was the meaning of ‘rate’ as the constant ratio of two continuously changing quantities, based on Thompson’s work (P. W. Thompson, 1990). The idea of constant ratio would have helped both in talking about quantitative changing smoothly, and in the discussion of equivalent rates. I instead focused my attention during this teaching episode on a meaning of rate as a proportional relationship between two changing quantities, using the model of talking about rate as a “fraction of the time means the same fraction of the amount.” I took this reasoning from other Thompson works (A. G. Thompson & Thompson, 1996; P. W. Thompson & Thompson, 1994). I did not notice the differences between these two ways of thinking about rate.

By neglecting continuous variation and constant ratio in my own thinking, my own meaning of rate was a chunky proportion, similar to one of Tiffany's meaning of rate, while all the time I was under the impression that I was thinking about rate continuously. My own chunky thinking about rate can be seen most clearly in my revision of the compound interest in the phase plane task, where I referred to rate as "the amount of interest earned during a very short period after that time expressed as a number of dollars per year." In the wording of this task, I equated rate with amount just as Tiffany did.

Per-capita Rate of Change

During the first teaching episode on simple interest, I managed to stump the students and myself by asking about the meaning of .08 in their function $q(x,n)=n+.08(n)x$. In the debriefing session, Pat suggested that reason this question might be problematic is because the .08 had no deep mathematical meaning—it was simply a bank policy for setting rates of growth as a way shaping the customer's behavior. Over the course of the teaching experiment, however, the students and I developed a number of meanings of .08 in various contexts.

8% as Chunky Multiplication

The student's initial interpretation of the simple interest model was as a model compounded annually (Chapter 4, Excerpt 1). In Tiffany's original explanation of the simple interest task, she describes a process of multiplying the current value of the account by 8% to find the change in the account over the next one-year chunk. Derek agreed with this interpretation. In this way of reasoning then, the 8% represents the factor by which Tiffany calculates the size of the next chunk. Had we continued with this meaning of 8%, the result would have been a geometric series with a growth factor of 1.08.

8% as a Policy

In the debriefing session after the simple interest episode (E1D1T1P1), Pat suggested a different interpretation of the 8%, that it was a policy used to determine the linear growth rates of customer's accounts as a way of shaping the customer's behavior. Thinking about 8% involves imagining that the simple interest account has a linear rate of growth, and that linear rate of growth is always a constant multiple of the initial invesement. (r=.08n). However Pat's insight into the meaning of 8% is deeper than this, because it also requires thinking about the 8% simple interest bank policy in a world of other possible bank policies, where other functions are used to set the rate.

In order to see the utility of relating the rate of change of the account proportionally to the initial investment, a student must first imagine a situation in which the rate of change is not proportional to the initial investment, for example, when the rate of change of any account's value in dollars per year is constant across all accounts. In such a situation (for example, if the bank offered growth of 8 dollars per year for every account regardless of initial investment), a customer could game the system to earn more money by breaking up their investment into multiple smaller accounts. The 8% policy is a reaction to this strategy. By offering to make rate proportional to initial investment, the bank accomplishes the task of treating all investments of $500 the same way, regardless of how many accounts the customer opens. The bank also prevents itself from inadvertently paying more money than it intended.

Per-capita Rate of Change

My current interpretation of the simple interest policy, is that 8% represents “8 cents on the dollar” or that every dollar in the initial investment earns .08 dollars per year. Or alternatively the 8% policy has two components: that a one dollar investment earns .08 dollars per year, and that an investment of any size has a rate in dollars per year proportional to .08 dollars per year. In this per-capita rate of change interpretation, .08 has units of “dollars per year per dollar.” And the “8” in “8%” has units of dollars per year per one hundred dollars. The interpretations of 8% as “8 cents on the dollar (or part thereof)” and as “.08n dollars per year” are connected by the distributive property of multiplication over addition. It is the difference between thinking of $100 dollars as a single number of dollars and thinking of $100 dollars as 100 individual dollars.

This meaning of per-capita rate of change (every dollar earns .08 dollars per year) is closely related to the meaning of per-capita rate of change I intended to use in the PD8 account task. In the original design of the PD8 account task, I described the account policy from an individual perspective: that every dollar *in the account* earns .08 dollars per year. This form of per-capita rate of change differs from the above in that the value earning interest: the number of dollars currently in the account in the account is changing continuously in time, and so the resulting .08n population rate is also changing continuously. In order to distinguish the two, I will refer to the rate accumulated from each *initial* individual as “simple per-capita rate of change.” And rate accumulated from each *current* individual as “recursive per-capita rate of change.”

However this meaning of recursive per-capita rate of change description I gave in my task was ambiguous. The students had a number of interpretations, depending on the student’s meaning of rate.

Chunky Per-Capita Rate of Change

Tiffany’s interpretation of recursive per-capita rate of change occurred within her chunky framework. At the beginning of the chunk, Tiffany imagined the rate of change of the account being calculated from the value at the time. Tiffany then calculated the value of the account at the end of the time chunk by using the rate of change of the account as a chunky proportion: imagining a completed change to get to the end of chunk as a fraction of the rate. At the end of the chunk the account then had a new value and the process repeated.

In this way, Tiffany generated a number of compound interest account behaviors, with the compounding interval tied to the size of the observational chunk that she was using. When I asked Tiffany about years, she imagined the account changing in year sized chunks and compounded annually. When asked Tiffany about hundredths of a second, she imagined the account changing in hundredth of a second sized chunks and compounded every hundredth of a second.

In imagining the time and account value changing in chunks, Tiffany did not imagine the account value changing within chunks. The evidence of this is that Tiffany only updated the rate of change every chunk (Chapter 4, Excerpt 19), when the PD8 task stated that the rate changed “every time your account balance changes.” Derek pointed out the ambiguity of this phrase.

Excerpt 78 -- Episode 6, 00:27:19

1 Derek: Does your account only change at the end of the year? Or do they calculate it all the way in between? ‘Cause it doesn't say.

Here Derek shows that he is thinking of a smooth rate, but also open to the possibility of a chunky rate. When Derek imagines the recursive per-capita rate of change as defining a population rate of change in a smooth context, he imagines a continuously changing account, and a continuously updated rate. However Derek is also open to the possibility of an account that does not update until the end of a chunk (year).

When thought about in terms of chunky changes to the account, the compounding interval is tied to the chunk size, making the compounding interval associated with recursive per-capita rate of change ambiguous. My suspicion is that the reason why I have received numerous objections that a recursive per-capita rate of change does not specify a compounding period is because the readers of the problem were imagining chunky changes.

A Rate of Change of a Rate of Change.

Another meaning of recursive per-capita rate of change used was per-capita rate of change as (the rate of change of (the rate of change of the population with respect to time) with respect to population). Symbolically, if P represents the population, then the rate of change of the population with respect to time could be represented as $r = \frac{dP}{dt}$ and the per-capita rate of change using this meaning would be $\frac{dr}{dP}$.

This meaning of per-capita rate of change is not consistent with the use of per-capita rate of change in mathematical biology, where per-capita rate of change is calculated as $\frac{r}{P}$.

This meaning of recursive per-capita rate of change was used when I asked Derek and Tiffany (separately) to find the slope of the PD8 financial model in the phase plane as well as the units of that slope. However, the most striking usage of this meaning comes from Derek's work with the food (Verhulst) model. In Figure 70, Derek has created a function for the per-capita rate of change y of the Verhulst model as a function of population. Derek created this function $\left(y = \left(\frac{-1/25}{70\,billion} \right) x + 1/15 \right)$ as result of imagining that the growth contribution of an individual would decrease with increased population (and implied increased competition for food) until the per-capita rate of change reached zero at carrying capacity at 70 billion people.

In the excerpt that follows, Derek is working with the equation for the per-capita rate of change as a function of population (Figure 70), and he describes how he would find the population rate of change as a function of population.

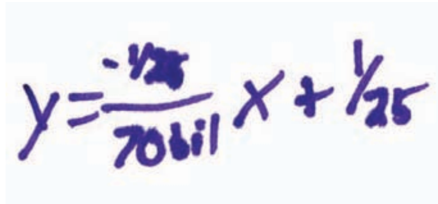


Figure 70. Derek's function for the food model. y is "the rate of growth for each [Kelly]." x is the population.

Excerpt 79 -- Episode 14, 0:48:24

1 Carlos: OK, so now what if I wanted to take this and produce, umm, a function that would predict the rate of growth in [Kellies] per year from the number of [Kellies].

2 Derek: So [Kellies] per year, It would you have to combine all of em.

3 Carlos: OK

4 Derek: Like you have to do it for every amount and then add them all.

In the excerpt above, Derek is proposing an integration, or a Riemann sum as a way of calculating the population rate of change function from the per-capita rate of change function, suggesting that Derek was imagining the per-capita rate of change as the rate of change of rate with respect to population $\left(\frac{dr}{dP}\right)$. Biologically, the integral approach works if one imagines that an individual, once born, commits to a specific growth rate and never changes that rate even as the population changes. However, in standard mathematical biology, the Verhulst model is one in which each individual is always adjusting their birth rate based on the available food (or population) at the moment, and is therefore using the $\frac{r}{P}$ meaning of per-capita rate of change.

Waiting Time

In the population context, Derek also used per-capita rate of change as the reciprocal waiting time for an individual to make another individual. Specifically, he imagined that a population growing at one Kelly per Kelly every 25 years would be a population in which a single Kelly would wait 25 years doing nothing and then instantaneously create a new whole Kelly. Thinking about per-capita rate of change in this way led Derek to imagine the rate of the population as the reciprocal of the waiting time for the next Kelly to appear. He imagined the waiting time decreasing as the population increased, and graphed a step function version of the exponential (Figure 58).

Constant Per-capita Rate of Change

The final (in this listing, not chronological) meaning of recursive per-capita rate of change that Derek used was the meaning of per-capita rate of change that I had originally intended in the design of the experiment: as the rate of change of the linear contribution that a single individual makes in a population of identical individuals. When Derek used this meaning, he imagined each Kelly or dollar in the population as producing offspring continuously at a constant rate. The proportionality of rate to amount in exponential growth then came from the accumulation of these individual constant rates across the population, as the population changed the number of individuals contributing at a constant rate increased, so the overall effect was that the rate of growth of the population increased proportionally to the increasing population.

Excerpt 80 -- Episode 13, 00:38:45

1 Carlos: OK. So can you explain to me what this population … what this model says – the … the model that we've created – about how the population of [Kellies] would grow.

2 Derek: It says that they grow by like … You have one [Kelly] and then another [Kelly] slowly appears like from bottom up. Starts at the feet and just grows, and then all of a sudden it's all there. And then you get more and more and then more just start popping up.

In this excerpt, Derek describes the effect of constant per-capita rate of change, firstly describing the continuous growth contribution of a single individual, he then switches to a waiting time meaning of per-capita rate of change. When he describes the population as a whole, "the more and more just start popping up" is referring to the Kellies coming in more and more frequently, and this frequency has a distinct discrete flavor to it. Here Derek is mixing discrete and continuous population images. He describes the population as growing continuously "another [Kelly] slowly appears." And also discreetly "and then all of a sudden it's all there." He also describes the rate of change changing discretely with a discrete population. In this description, more Kellies do not start "popping up" until after the first offspring is complete. What is missing from this description is the idea that as a Kelly is producing new Kellies continuously, the fractions of Kellies being produced are also producing new Kellies, an omission which I did not catch during the teaching episode itself.

Exponential Growth

The meanings of exponential growth that I will discuss here are the meanings that the students walked away with at the end of the teaching experiment: specifically, the results of the phase plane analysis task, in which the student constructed a graph of the PD8 financial account over time from a linear phase plane graph, and Derek's work with the Malthus model. Overall, the students walked away with three understandings of exponential growth as a result of this teaching experiment. These meanings are very closely tied to the student's understandings of rate of change and per-capita rate of change.

Tiffany's Exponential

Tiffany's image of exponential growth was based on thinking in chunks, particularly Tiffany's image of rate proportional to amount in a chunky context. Tiffany imagined the exponential as a sequence of points (Figure 71), with the calculation to find each point based on the previous point. In order to find the next point in the sequence, Tiffany imagined time changing in chunks of uniform size. For each chunk of time, the corresponding change in account value was based on two ideas: that the 'rate' (amount earned in a year without compounding) was calculated as .08 times the current value of the account, and that the change from one point to the next could be found by applying chunky proportionality to find the 'rate' as amount for her observational time chunk.

Because Tiffany did not imagine change occurring within the chunks, Tiffany plotted each point individually, and Tiffany's resulting function was a function of compounded growth, with the compounding interval equal to the size of the observational time chunk.

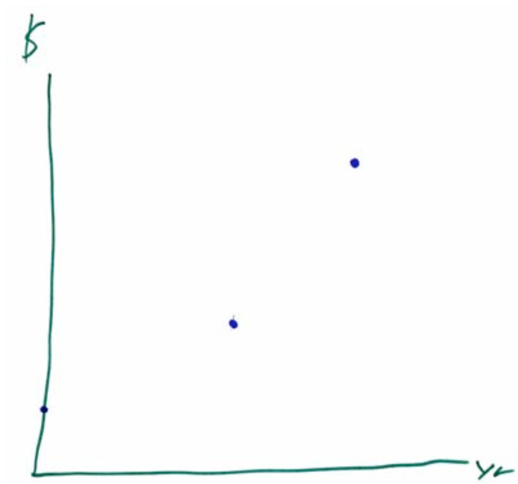


Figure 71. Tiffany's graph of the value of the PD8 account over time for the first two seconds.

Tiffany was not aware that changing the observational unit (and thus the compounding period) would change values of the function that she had already calculated. Tiffany did not have a strong sense of the behavior within each time chunk, whether it be linear, curved, a series of more closely spaced points, or nothing at all.

Overall, Tiffany never reached an understanding of exponential growth that was separate from discrete compounding, and she never reached an understanding of exponential growth as a functional relationship from time to account value. Although Tiffany could use compounding to calculate values of compound interest accounts, she imagined each account value as being calculated from the previous account value, not from the current time. Tiffany imagined that she could calculate the value of an account at any specific time, but she never built a relationship in which the value of the account was dependent on the value of time.

Derek's Continuous Model

Derek had two models of exponential growth, depending on whether or not he imagined that the population could take on continuous values. In the case of the financial model, Derek used the idea of rate "how fast" proportional to a continuously changing account value to imagine that the account was growing "faster and faster." When imagining the population continuously, Derek drew exponential growth as a curve that was increasing as an increasing rate (Figure 72).

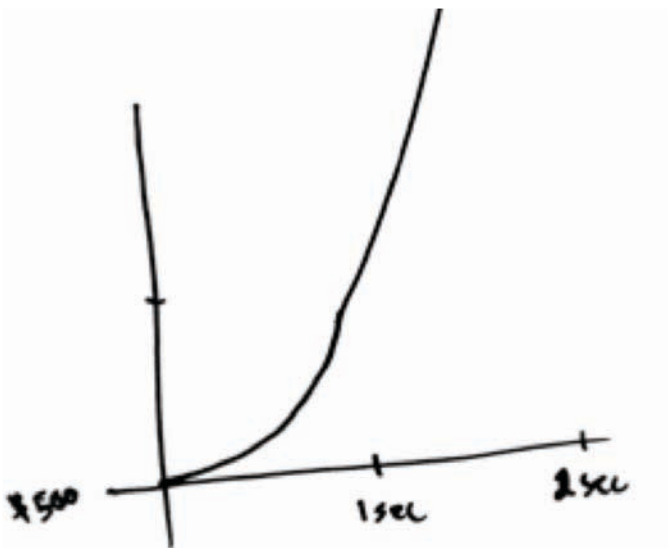


Figure 72. Derek's graph of the PD8 account value over time (cropped).

Although Derek derived this continuously growing model in part from reasoning about individual dollars and per-capita rate of change, it is impossible to determine if Derek's image of per-capita rate of change in the financial context was based in per-capita rate of change as a rate of change of a rate of change ($\frac{dr}{dP}$) or based in per-capita rate of change as a constant growth contribution of an individual ($\frac{r}{P}$), or if Derek made a distinction between these two ways of thinking at all. In an exponential context, when per-capita rate of change is a constant value, these ways thinking produce identical results.

Derek's Discrete Population Model

Derek's other model of exponential growth was based on rate of change as the reciprocal of waiting time. In this model, as the population increased, the rate increased, and the waiting time for the next Kelly (or cent) to be created decreased. Derek imagined the population growing in steps over time, with the length of each step representing the decreasing waiting time for the next Kelly to be born, and the height of each step being a change in the population of 1 (Figure 69)

Although Derek did not graph this function carefully, he described 1 Kelly as having a waiting time of 25 years and 25 Kellies as having a waiting time of one year, establishing an inversely proportional relationship between population and waiting time. If we calculate slightly further than Derek did, we see that Derek would have predicted that one Kelly has a waiting time of 25 years, two Kellies a waiting time of 25/2 years, and P kellies a waiting time of $w = \frac{25}{P}$. Since Kellies can only increase one Kelly at a time, total waiting time to grow from 1 to P Kellies would be based on the harmonic series. So although Derek did not push it this far, we can think of the step function that Derek drew as the inverse of a harmonic series.

Contrasting Derek and Tiffany

Derek and Tiffany were very different in the flexibility with which they adopted different ways of thinking. Tiffany adopted an approach that was entirely self-contained and self-consistent, while Derek adopted a number of different perspectives as he needed them. Not all of Derek's perspectives were consistent with each other.

With one or two possible exceptions, Tiffany never engaged in smooth thinking. She only used chunky thinking. The meanings that she developed during the teaching experiment were all consistent with this chunky thinking. She only had two meanings of rate: rate as amount, and rate as chunky proportion. She only used two meanings of per-capita rate of change: 8% as chunky multiplication and chunky per-capita rate of change. Finally, she only developed one understanding of exponential growth: the exponential as chunky compound interest.

Derek on the other hand, adopted a broad variety of perspectives as needed. He made use of both chunky and smooth thinking, and switched between them freely as needed. He used four meanings of rate on the list: rate as amount, rate as chunky proportion, rate as an index of "how fast," and rate as a waiting time. In interpreting the meaning of 0.08, Derek used 8% as chunky multiplication, rate of change of a rate of change, chunky per-capita rate of change, constant per-capita rate of change, and individual waiting time. Derek developed two understandings of exponential growth: his continuous exponential model and his discrete population model, in addition to an understanding of compound interest. These many different ways of thinking cannot be attributed to the change in model context. Some of these ways of thinking are more clearly identified in the biological context, but Derek used all of these ways of thinking during the financial model portion of the teaching experiment, before any biological context was introduced.

Not all of Derek's meanings were consistent. Rate as waiting time, for example, predicts different values for the related quantities than rate as chunky proportion does, and Derek's exponential models predict different population values. However, all of Derek's meanings were compatible. Derek navigated through this multiplicity of meanings fluently. He used these different inconsistent ways of thinking to build off of each other. He coordinated different ways of thinking in mutually supportive ways that advanced his understanding of the problem at hand. He used the conclusions from one perspective to advance his understanding when he switched to a second or third perspective. Sometimes Derek was aware of the shifts in thinking that he was making (such as when he distinguished between discrete and continuous populations), but frequently Derek was unaware of the differences between the ways of thinking that he was using. Derek's myriad of meanings of rate were all 'the rate.'

Despite – or perhaps because of – this myriad of meanings that Derek had, Derek did not misstep in his shifting ways of thinking. He always chose the correct tool for the job, except in situations where – as a result of my ambiguous wording – the job was unclear. I have no idea what the operational mechanism by which Derek selected any particular appropriate way of thinking might be. Since I did not identify these different ways of thinking until retrospective analysis, it is impossible to design tasks and questions to reveal how Derek shifts between them.

I can, however, speculate on the general reason for Derek's flexibility. Derek imagined every situation changing in continuous time. By imagining time changing continuously over an interval, Derek could also keep track of the starting and ending states of the situation. In imagining every state between the beginning and ending states, Derek also imagined the beginning and ending states. This image enabled Derek to use smooth thinking as an origin for chunky thinking. So Derek could view the same continuous time situation through multiple lenses. Tiffany, in contrast, only imagined individual states, rather than intervals. Since imagining only the beginning and ending states is not sufficient to specify intermediate states, Tiffany only engaged in chunky thinking.

CHAPTER 7

ANALYSIS OF THE DESIGN

Through the lens of “chunky” and “smooth” the original design of the teaching experiment appears very different to me now than it did when I designed it. The teaching experiment, and the tasks that comprised it were based on a blending of perspectives of the exponential, all filtered through my own chunky lens. In order to discuss this blending of perspectives, I must revisit the work of Thompson (Saldanha & Thompson, 1998; Thompson, 1990, 2008a) and Confrey and Smith (Confrey & Smith, 1994, 1995), and specifically how my understanding of these authors’ works has been altered by the distinction between completed change and change in progress. Where I once interpreted Confrey and Smith’s work as discrete variation, and Thompson’s work as continuous variation, I now interpret Confrey and Smith’s work as chunky reasoning, my prior interpretation of Thompson’s work as chunky reasoning, and my current interpretation of Thompson’s work as describing a mixture of chunky and smooth reasoning.

Confrey and Smith

In my original interpretation of Confrey and Smith (Chapter 2), I interpreted the main vehicle of Confrey and Smith’s style of variation to be iteration. Confrey and Smith propose two classes of “number worlds” based on the arithmetic sequence and the geometric sequence. An additive or counting world is based on an arithmetic sequence, where successive elements have a constant difference, and a multiplicative or splitting world is based on a geometric sequence, in where successive elements have a constant multiple (Confrey & Smith, 1994; Smith & Confrey, 1994). Thus each type of sequence has a successor operation, adding a constant value in the case of an arithmetic sequence (+c), or multiplying a constant value in the case of a geometric sequence (*n). I originally interpreted these successor operations as describing iterative processes. However I now see Confrey and Smith as describing unit-based chunky thinking, in which reasoning arises from iteration, but is not constrained by iteration.

The Role of Confrey and Smith's "Rate" in the Teaching Experiment

Confrey and Smith use two meanings of rate in their work. One meaning of rate is the meaning of rate traditionally used in calculus (Smith & Confrey, 1994). In this paper, Smith and Confrey described the exponential movement of a point using the differential equation form of the exponential. This is the meaning of rate that I had in my own mind during the planning and the initial few episodes of the teaching episodes. As I was called to formalize this understanding in a way that I could communicate to Derek and Tiffany, my image of rate became more and more like Confrey and Smith's second meaning.

The second meaning of rate described by Confrey and Smith (Confrey & Smith, 1994, 1995) is the meaning that has the closest relationship with this teaching experiment. This second meaning of rate depends on their notion of 'unit,' which is defined as "a repeated action" (Confrey & Smith, 1994). Thus in an arithmetic sequence, there is a repeated action of adding c, so +c is the repeated action. Confrey and Smith (1994) represent this unit as Δc. Similarly a geometric sequence is generated by the repeated action of multiplying by n. Confrey and Smith (1994) represent this unit as ®n. These repeated actions of uniform size are very similar to Tiffany's chunky variation of time.

Confrey and Smith (1994) define rate as a unit-per-unit comparison. That is, for each pair of coordinated sequences, the repeated actions may be compared to form a rate. Thus a function generated by a multiplicative sequence of unit ®4 and an additive unit of Δ3 has a multiplicative-additive rate of ®4:Δ3. This gives rise to potentially four meanings of rate: an additive-additive rate (constant for linear functions), a multiplicative-additive rate (constant for exponential functions), a multiplicative-additive rate (constant for logarithmic functions), and a multiplicative-multiplicative rate (constant for monomials).

My original interpretation of this meaning of rate falls under the category of "Chunky rate as amount," in that I understood Confrey and Smith as describing completed changes. I interpreted a rate of ®4:Δ3 as meaning a completed additive change of 3 and a completed multiplicative change of 4. However my current interpretation is that Confrey and Smith describe something very similar to "rate as chunky proportion" by developing isomorphic partitive operations that operate on additive and multiplicative units.

Beginning from arithmetic and geometric sequences, which have values only over whole numbered indices, these sequences can be extended to rational valued indices by considering the effect of composing the successor operation, and reversing the process to create a partitive operation. Since the effect of repeated *c*-counts is multiplication, then the partitive operation would be division, and the value of an arithmetic sequences rational indices can be found by arithmetic mean (Confrey & Smith, 1995). Since the effect of repeated n-splits is exponentiation, then the partitive operation would be nrooting, and the value of geometric sequences at rational indices can be found by the geometric mean (Confrey & Smith, 1995; Strom, 2008).

The rational indexed arithmetic sequence is constructed by a number of actions: the action of finding a successor by adding c, the action of composing successor actions, the action of adding jc that results from making j compositions, the action of partitioning the successor action into j actions that would compose to adding c, and the resulting partitioned successor action of adding $\frac{c}{j}$.

Confrey and Smith (Confrey & Smith, 1995) argue that each of these actions has an isomorphic action in a multiplicative world of a geometric sequence: the action of finding a successor by multiplying by n, the action of composing successor actions, the action of multiplying jn that results from making j compositions, the action of partitioning the successor action into j actions that would compos to multiplying by n, and the resulting partitioned action of multiplying by $\sqrt[j]{n}$.

Although to my knowledge Confrey and Smith did not state it, these partitive operations allow the creation of equivalent rates. By making use of the isomorphism between partitive operations, equivalence classes of rates can be developed, allowing for rate to have a meaning that is independent of action size. A coordination of a repeated action of multiplying by 4 and a repeated action of adding 3 is also a coordination of a repeated action of multiplying by $\sqrt[7]{4}$ and a repeated action of adding $\frac{3}{7}$. Thus all multiplicative additive rates of the form $® \sqrt[j]{n} : \Delta \frac{c}{j}$ are equivalent to the multiplicative additive rate ®$n : \Delta c$. This meaning of rate is similar to "rate as chunky proportion" in that both partition rate as amount in the same way to generate equivalent additive-additive rates, but differs from "rate as chunky proportion" in that the "proportionality" is extended to multiplicative units as well.

Thompson's Continuous Variation

My original interpretation of Thompson's continuous variation and continuous rate was through a chunky lens. I described Thompson's (2008b) meaning of continuous variation as imagining change occurring over intervals of conceptual time, and imagining that within each interval the quantity assumes all intermediate values by repeating this process recursively over sub-intervals of the original interval. While this is the process that Thompson describes, my interpretation of this process focused on the recursive process of intervals and sub-intervals, rather than on assuming all intermediate values. This is reflected in my understanding of Thompson's meaning of speed as "in any fraction or multiple of that time, the person traveled an equal fraction or multiple of the distance," a clear example of a rate as chunky proportion.

This interpretation of Thompson's work, describing changes occurring in intervals and rate as chunky proportion is correct. Thompson and his collaborators described continuous variation as occurring in completed changes over intervals (Saldanha & Thompson, 1998; Thompson, 2008b) and speed as a chunky proportion (Thompson & Thompson, 1994). However these examples of chunky reasoning in Thompson's work do not necessarily form the whole story.

> An operative image of covariation is one in which a person imagines both quantities having been tracked for some duration, with the entailing correspondence being an emergent property of the image. In the case of continuous covariation, one understands that if either quantity has different values at different times, it changed from one to another by assuming all intermediate values (Saldanha & Thompson, 1998, p. 2).

In the above excerpt, Saldanha and Thompson appear to be talking about completed changes: two quantities have been tracked for some duration, and this would be an example of chunky reasoning. However, the description of covariation by Saldanha and Thompson leaves open the possibility of using smooth reasoning as well. If a person imagines quantities "having been tracked for some duration," one possible interpretation is the person has imagined the tracking, rather than the person imagined that the quantities have been tracked. In the first interpretation, there is smooth reasoning in imagining the tracking process in experiential time, and chunky reasoning in thinking about the tracking process after it has been completed. Similarly, there is room for smooth thinking in how the person imagines "assuming all intermediate values" which could occur in experiential time rather than abstractly.

If one imagines "assuming all intermediate values" as taking place in experiential time, then Saldanha and Thompson's (Saldanha & Thompson, 1998; Thompson, 2008b) meaning of continuous variation takes on a very different character, in that the origin of the continuity is different. In my original interpretation of Thompson's (2008b) meaning of continuous variation, I imagined continuity as arising at each interval, with the real numbers between the endpoints, and variation arising from the coordination of the endpoints of the intervals and subintervals. In my current interpretation of Thompson's work, I understand each interval as imagined as a completed "chunk," and variation within an interval is imagined by two processes: imagining the smooth variation from endpoint to endpoint in continuous experimental time (assuming all the values in between) and imagining that the interval is also cut up into chunky sub-intervals, each of which contains smooth variation.

A clear example of the juxtaposition of chunky and smooth perspectives in Thompson's work is seen in the interpretation of the difference quotient that he describes as a "sliding secant" (Thompson, 1994). Thompson describes the function $f_h(x) = \dfrac{f(x+h) - f(x)}{h}$ as giving the slopes of secant lines. He imagines that for a fixed value of *h*, each value of *x* describes a secant line between the points (x, f(x)) and (x+h, f(x+h)), and that as an interval of length *h* is imagined sliding through the domain of *f*, the function f_h keeps track of the slope of the secant defined by that sliding interval. By imagining a "sliding secant" Thompson is using both chunky and smooth reasoning simultaneously. Chunky reasoning is used in the fixed interval of size *h*, and in the completed changes f(x+h)-f(x) and x+h-x defined by that interval, while smooth reasoning is uses in imagining the interval, the secant, and its slope all sliding in experiential time.

An example more relevant to this teaching experiment is Thompson's two meanings of rate of change: as a constant ratio between smoothly changing quantities (Thompson, 1990), and as a chunky proportion in which a fraction of the (completed) time chunk means the same fraction of the (completed) amount chunk (Thompson &

Thompson, 1994). Mathematically, these two meanings of rate are equivalent, so long as "fractions" are permitted to be any ratio of two real numbers. Conceptually, these two meanings of rate are very different. This can be seen in the difficulty that all four participants in the teaching experiment had in justifying why equivalent rates are equivalent using rate as a chunky proportion. Justifying the equivalence of two rates using rate as constant ratio is trivial. In this teaching experiment, my use of rate as chunky proportion to the exclusion of rate as constant ratio came from my reading of Thompson through an exclusively chunky lens.

The Design of the Tasks

The design of the tasks came from two sources: my chunky understanding of Thompsons' construction of the exponential and my own smooth image of "constant per-capita rate of change." Much of the confusion of the teaching experiment, on the part of the students, myself, and my witness, can be attributed to my inadvertently attempting to design an experiment that taught smooth concepts from a chunky perspective. Examples of this include the "rate as amount of interest…" clause in the compound interest phase plane task (Episodes 7-9), the "SayCo's new feature..." clause of the PD8 account policy (per-capita rate of change task, Episode 6), and the numerical analysis approach of fixing a rate over a small interval of time to estimate exponential growth in the phase plane (Phase Plane Analysis task).

This confusion of perspectives created an environment where a large number of meanings of rate and a large number of understandings of the exponential function family could flourish. Not all of these meanings were compatible with each other, and not all of these meanings were compatible with my goals, either my stated chunky perspective goals (Tiffany's exponential), or the smooth perspective goals (Derek's continuous exponential) that I was not aware of until the end of the experiment.

More importantly, because I questioned the students entirely from a chunky perspective, I did not put any pressure on Tiffany to adopt a smooth perspective. Even in situations where I believed that I was pressing Tiffany to think about continuous variation, the suggestions and questions that I used were all compatible with Tiffany's chunky perspective. In this environment, Tiffany never developed multiple ways of imagining change, rate, per-capita rate, and exponential growth as Derek did. Instead, Tiffany was able to use her chunky perspective throughout the entire teaching experiment. I have no idea if pressuring Tiffany to use smooth thinking would have resulted in the flexibility and fluency that Derek had. I suspect that such flexibility would not have come so easily to Tiffany as it did to Derek. However, I can be certain that – as a result of my own chunky perspective – the opportunity to investigate the effect that smooth thinking would have on Tiffany's mathematics never arose.

CHAPTER 8

CONCLUSION

In the analysis of this teaching experiment, I identified two different ways of thinking about change, chunky and smooth, and five different ways of understanding exponential growth: the chunky approach of Confrey and Smith and Strom (Confrey & Smith, 1994, 1995; Strom, 2008), the approach of mixing smooth and chunky reasoning used by Thompson (Thompson, 2008b), my own thinking on constant per-capita rate of change in a smooth context, Derek's idea of waiting time, and a stochastic model of exponential growth not previously discussed. I will discuss each of these understandings in turn.

Chunky and Smooth

Change can be imagined in two different ways, by imagining changes that have been completed (chunky), or by imagining a change in progress (smooth). The key distinction between the two ways of reasoning is whether or not the change itself takes place in experiential time. Chunky completed changes can appear to be smooth changes in progress when the student imagines that chunks are strung together in sequence. Tiffany's description of an account changing "little by little" is an example of this distinction. Although Tiffany saw the changes taking place sequentially in experiential time, she did not imagine each individual change taking place in experiential time, which is the mark of true smooth thinking.

The Geometric Exponential

The exponential as developed by Confrey and Smith and Strom (Confrey & Smith, 1994, 1995; Strom, 2008), is an extension of a geometric sequence to rational indices by means of an externally imposed mathematical convention. Imagine that we have a whole geometric sequence S, with whole indexed terms S_i defined by repeated multiplication $S_i = nS_{i-1}$. The mathematical convention suggested by these authors is that the value of the rational indexed term $S_{i+\frac{p}{q}}$ be calculated as $\left(\sqrt[q]{n}\right)^p S_i$. This way of understanding the exponential is chunky because it keeps track only of completed multiplicative and additive changes. The authors justify the convention by partitioning completed changes into smaller completed changes using an extension of rate as chunky proportion to multiplicative units.

The Compound Exponential

The exponential as described by Thompson (2008a) is based on a compound interest model using a combination of chunky and smooth reasoning. Thompson's construction requires that the student attend both to the behavior at each compounding point (chunky thinking) and by imagining linear growth in-between compounding points by means of smooth thinking.

Thompson's (2008a) description of the construction of the exponential begins with simple interest where the growth of an investment account is proportional to time, and the growth rate of the account is proportional to the initial investment. However, from the debriefing sessions in which Pat described his thinking of 8% as policy, it is clear that simple interest is not the beginning of this chain of reasoning. Thompson's understanding of simple interest comes as a reaction to a policy of fixed rate linear interest (for example, 8 cents per year, regardless of initial investment), which can be abused by investing one's money in multiple accounts.

Thompson's simple interest policy describes two rates: the proportional relationship between changes in the account and changes in time, and the proportional relationship between the rate of growth of the account and the initial value of the account. The equation below demonstrates these two proportional relationships in a single function with a fixed principal of $1000, and a fixed interest accumulation rate of 1.08*1000 dollars/year.

$$y = 0.08(1000)x + 1000 = 1.08(1000)x \tag{19}$$

Now, imagine that that interest is compounded at the end of every year. Two things change in this new model. The interest rate–originally, the proportional relationship between rate of growth and the *initial* value of the account–is re-conceptualized as a proportional relationship between the rate of growth and the value of the of the account at the beginning of each year. While the rate of growth–the proportional relationship between the accumulation of dollars and the accumulation of years--changes at the end of every year. This results in a piecewise linear growth function.

As the size of compounding interval decreases, two things happen. Firstly, the piecewise linear function begins to have more and more bends in it, "smoothing out" until in the limit it becomes a curve consisting entirely of bends. Secondly, the compounding points become more and more frequent, until we imagine that–in the limit–

the compounding points are the entirety of the function, so the property that the rate of growth of the function is proportional to the value of the function at the beginning of a compounding interval becomes the property that the rate of growth of the function is proportional to the value of the function all the time: the property of exponential growth used in differential equations modeling.

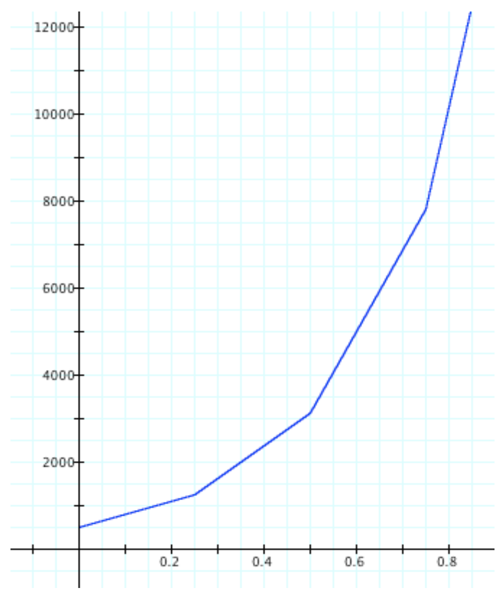


Figure 73. Value of a compound interest account over time, with parameters chosen to exaggerate the piecewise linearity of the growth: Initial investment P = 500, interest rate r = 600%, compounded four times annually.

Thompson's compound exponential is incompatible with Confrey and Smith's geometric exponential because they make different modeling assumptions. Confrey and Smith's assumption that intermediate values can be calculated by geometric means does not hold for Thompsons' piecewise linear compound interest functions, where intermediate values would need to be calculated using arithmetic means (if using Confrey and Smith's toolkit).

The Phase Plane Exponential

Thompson's approach begins by taking rate proportional to initial investment as a policy, and by exploring the consequences of such a policy, derives a policy in which rate is proportional to current amount as a result. The phase plane approach, used in mathematical biology (Bergon, Harper, & Townsend, 1996; Brauer & Castillo-Chavez, 2000; Edelstein-Keshet, 1988), begins by taking rate proportional to current amount as an assumption of the model.

Rate proportional to current amount can be justified as an assumption in several ways. One way is to imagine the growth of the population at the population level: if a certain population N has a certain associated rate of change r, then it is not an unreasonable assumption that a population pN would have an associated rate pr. A second way to justify rate proportional to current amount is at the individual level, by imagining constant per-capita rate of change. If every individual contributes to the population at a (smooth) constant per-capita rate r, then the entire population N would have a population rate of change of Nr. This second meaning was the original intent of my per-capita PD8 account policy.

The phase plane exponential differs from the compound exponential and the geometric exponential in two primary ways: first, that the reasoning needed in order to interpret the problem and reach a solution must be entirely smooth, as demonstrated by Derek. Second, the understanding of exponential behavior reached by this smooth thinking is entirely qualitative. Derek's graph of the growth of the population over time has no numbers on the axes for good reason. The tools of Calculus are needed to reach an analytical solution to the behavior of the function described in the phase plane. This makes a particular sort of sense if we imagine that smooth reasoning is always varying continuously, and so always involves infinitesimals. Calculus is the method by which mathematicians quantify smooth reasoning.

Even with Calculus as a tool, qualitative analysis still plays a huge role in dynamical systems. As differential equation models become more complex, qualitative results are often the only results that applied mathematicians are able to practically reach. A common approach to complex systems of differential equations is to describe qualitatively all possible classes of behavior of the system, rather than solve for the functions that determine that behavior explicitly at every point in time.

The Harmonic Exponential

Derek's image of rate as describing a waiting time until the next whole individual was added to the population resulted in a fourth distinct way of imagining exponential growth. Derek's approach also used rate proportional to current amount as a base assumption, but Derek imagined a population growing discretely in continuous time. Specifically, Derek imagined that if the waiting time for one individual to produce another individual was $\frac{1}{r}$, then the waiting time for N individuals to produce another single individual would be $\frac{1}{rN}$. Once Derek graphed this behavior as a step function he was finished with this way of understanding the exponential, but we can push this idea a bit further.

Based on Derek's waiting time assumptions, the waiting time for a population of one individual to reach a population N would then be calculated by harmonic series

$\frac{1}{r}\sum_{j=1}^{N}\frac{1}{j}$. Since the limit of the harmonic series is approximately logarithmic, plotting each population value with its associated waiting time would result in a function that is approximately exponential.

Although Derek used chunky reasoning when thinking of waiting times (evidenced by his choice of a step function rather than independent points), this way of reasoning about the exponential could make use of chunky or smooth thinking equally well. The harmonic exponential differs from the other ways of understanding the exponential in that it takes an inverse perspective, using fixed increments in population to determine time, rather than taking time as an independent variable.

The Stochastic Exponential

Derek's 'waiting time' thinking is related to a fifth way of understanding exponential growth that I have not yet discussed. In stochastic processes, the exponential function is the expected value of a continuous time Markov chain where the waiting time from state N to state N+1 (representing population) is exponentially distributed with parameter $N\lambda$. This process is similar to Derek's in that it involves imagining the waiting time until the population increases by one. The stochastic exponential depends on an understanding of the expoential distribution, which is itself defined in terms of the exponential function. So any usage of the stochastic exponential to introduce exponential functions would be circular.

Multilingualism

All five of these models arrive at exponential behavior, but each of the models begins with different, contradictory assumptions. The harmonic and stochastic exponentials begin by assuming a discrete population, while the geometric, compound, and phase plane exponentials assume a continuous population. The geometric exponential assumes that intermediate values are calculated by the geometric mean; while the compound exponential assumes intermediate values grow linearly. The phase plane exponential begins with the assumption that rate is proportional to current amount, while the compound exponential begins with a policy of rate proportional to initial amount. Within the discrete population models, the harmonic exponential assumes deterministic waiting times, while the stochastic model assumes random waiting times.

All five of these models produce different conclusions. The phase plane model generates the smooth qualitative behavior of the exponential. The compound interest model generates rate proportional to current amount, and the value of e as the $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. The geometric model generates values of exponential functions for rational inputs. The stochastic model generates a distribution of possible population behaviors, and the harmonic model is one method of estimating the growth of the stochastic model.

I posit that professional mathematicians–in industry, academia, or education–understand exponential growth by making use of several of these models simultaneously. I believe that mathematicians who are successful in making sense of and using exponential growth switch rapidly between ways of understanding the exponential without being aware of it. These mathematicians adjust their understanding to the point of view that is most useful at the moment in order to make sense of the model that they are using, or to reach the type of conclusion that they want.

Derek provides an excellent example of this type of reasoning. Over the course of the teaching experiment, Derek made use of both chunky and smooth thinking. He imagined rate as chunky proportion, as “how fast,” and as the reciprocal of waiting time. He imagined per-capita rate of change as a rate of change of a rate of change, as an individual contribution, and as a waiting time. These changes in perspective allowed Derek to reach a much better understanding of exponential growth. At the same time, aside from the distinction between discrete and continuous population, Derek never gave any indication that he was aware of his changes of perspective.

In linguistics, the term “code-switching” refers to the use of more than one language in a conversation by a single speaker. Multilinguals find themselves in situations where one language or another is better suited to a particular topic, meaning, or social circumstance, and adjust their language choice to fit the needs of the conversation in the moment, even in mid sentence. Coming from a bilingual family myself, I am intimately familiar with the phenomenon. Often times, code-switching is so natural to a fluent multilingual that they can switch languages without being aware of the shift.

By analogy, I am proposing that mathematicians code-switch between ways of understanding exponential behavior. They make these switches rapidly–adapting to the model that best serves their needs in the moment–often without being aware that they have changed perspective at all. At the moment, the code-switching analogy is only a hypothesis. In identifying multiple ways of thinking about exponential growth, and studying Derek’s and my own shifting perspectives during the course of the teaching experiment, it seems reasonable that other mathematicians would engage in the same sort of shifting perspective reasoning. Confirming that this is the case is beyond the scope of this study.

If mathematicians do in fact code-switch between multiple ways of thinking of exponential growth, then that raises new questions for both research and instruction. By what mechanism(s) do mathematicians change between ways of thinking? Do teachers make use of the same set of ways of thinking about exponential growth that academic or industrial mathematicians do? Do these three groups use the same ways of thinking with the same frequency? How many ways of thinking about exponential growth are there really?

In the realm of instruction: How do we teach multilingual mathematical fluency to students in schools? Which different ways of thinking do we want students to learn? How do we insure that students with an insufficient number of ways of thinking develop additional ways of thinking? Is the current system of making no distinctions between different ways of understanding the best way to teach exponential growth? Should different ways of thinking exponential growth be taught concurrently, but with the distinctions made explicit? Would it be better to teach each way of thinking about exponential growth independently and sequentially? All of these questions have yet to be answered.

While the need for further research is extensive, the study of Derek and Tiffany has provided me with insights that serve as a solid starting point. Derek and Tiffany's different ways of thinking about change, covariation, rate, per-capita rate, and exponential growth show how inter-related these ways of thinking are. Derek and Tiffany built very different systems of reasoning from their different smooth and chunky ways of reasoning. Similarly, each of the ways of thinking about exponential growth discussed in this conclusion is an inter-related system of ways of thinking about change, rate of change, and exponential growth. Future work must take into account both the relationships between ways of thinking about the components of exponential growth, and the relationships between multiple systems of thinking about exponential growth itself.

REFERENCES

Ang, K. C. (2004). A simple model for a SARS epidemic. *Teaching Mathematics and its Applications, 23*(4), 181-188.

Barker, W., Bressoud, D., Epp, S., Ganter, S., Haver, B., & Pollatsek, H. (2004). *Undergraduate Programs and Courses in the Mathematical Sciences: CUPM Curriculum Guide 2004*. Washington, DC: The Mathematical Association of America.

Barker, W., & Ganter, S. (2004a). A Collective Vision: Voices of Partner Disciplines. In W. Barker & S. Ganter (Eds.), *Curriculum Foundations Project: Voices of the Partner Disciplines*. Washington, DC: Mathematical Association of America.

Barker, W., & Ganter, S. (Eds.). (2004b). *Curriculum Foundations Project: Voices of the Partner Disciplines*. Washington, DC: Mathematical Association of America.

Bergon, M., Harper, J. L., & Townsend, C. R. (1996). *Ecology: Individuals, Populations and Communities* (3rd ed.): Blackwell Science.

Berry, J. (2002). Developing mathematical modeling skills: The role of CAS. *ZDM, 34*(5).

Borba, M. C., & Villareal, M. E. (2005). Modeling and media in action *Humans with Media and the Reorganization of Mathematical Thinking* (pp. 101-120). New York: Springer.

Brauer, F., & Castillo-Chavez, C. (2000). *Mathematical Models in Population Biology and Epidemiology*. New York: Springer.

Burghes, D. (1980). Mathematical modelling: A positive direction for the teaching of applications of mathematics at School. *Educational Studies in Mathematics, 11*(1), 113-131.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: a framework and a study. *Journal for Research in Mathematics Education, 33*(5), 352-278.

Confrey, J. (1988). *Multiplication and splitting: Their role in understanding exponential functions.* Paper presented at the Annual Meeting of the North American Chapter for the International Group for the Psychology of Mathematics Education, Dekalb, Illinois.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit *Educational Studies in Mathematics, 26*(2), 135-164.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Edelstein-Keshet, L. (1988). *Mathematical Models in Biology* (First ed.). New Yrok: McGraw-Hill.

Gabriel, J., Saucy, F., & Bersier, L. (2005). Paradoxes in the logistic equation? *Ecological Modelling, 185*, 147-151.

Glasersfeld, E. v. (1995). *Radical constructivism: A way of knowing and learning*. London: Falmer Press.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolizing modeling and instructional design. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating in mathematics classrooms* (pp. 225-273). Mahwah, NJ: Lawrence Erlbaum Associates.

Harding, A. T. (1993). Review: Mathematics in medicine and the life sciences by F. C.Hoppensteadt. *The Mathematical Gazette, 77*(479), 271-273.

Hestenes, D. (2006). *Notes on a modeling theory of science, cognition, and instruction.* Paper presented at the GIREP Conference: Modeling in Physics and Physics Education, Amsterdam, Netherlands.

Huff, T. F., & Terrell, W. J. (2004). Health Related Life Sciences. In W. Barker & S. Ganter (Eds.), *Curriculum Foundations Project: Voices of Partner Disciplines*. Washington, DC: Mathematical Association of America.

Iovinelli, R. (1997). Activities for the logistic growth model. *Mathematics Teacher, 90*, 588-597.

Jones, M. S. (1997). Teaching mathematical modeling. *International journal of mathematics education in science and technology, 28*(4), 553-560.

Kingsland, S. (1982). The refractory model: The logistic curve and the history of population ecology. *The Quarterly Review of Biology, 57*(March), 29-57.

Lesh, R., & Doerr, H. (2003). Foundations of a models and modeling perspective on mathematics teaching, learning, and problem solving. In R. Lesh & H.

Doerr (Eds.), *Beyond constructivism* (pp. 3-33). Mahwah, NJ: Lawrence Erlbaum.

Lingefjärd, T. (2006). Faces of mathematical modeling. *ZDM, 38*(2).

Malthus, T. R. (1826). *An essay on the principle of population* (Sixth ed.). London: John Murray.

NCTM (2000). Principles and Standards for School Mathematics. Reston, VA: National Council of Teachers of Mathematics.

Pearl, R., & Reed, L. J. (1977). On the rate of growth of the population of the United States since 1790 and its mathematical representation. In D. Smith & N. Keyfitz (Eds.), *Mathematical demography: Selected papers* (pp. 341-347). Berlin: Springer-Verlag.

Rasmussen, C. L. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*(1), 55-87.

Rizzuti, J. M. (1991). *Students' conceptualizations of mathematical functions: The effects of a pedagogical approach involving multiple representations.* Unpublished doctoral dissertation, Cornell University.

Saldanha, L., & Thompson, P. W. (1998). *Re-thinking co-variation from a quantitiative perspective: Simultaneous continuous variation.* Paper presented at the Annual Meeting of the Psychology of Mathematics Education, Raleigh, NC.

Shoenfeld, A. H. (1991). On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In J. Voss, D. Perkins & J. Segal (Eds.), *Informal Reasoning and Education* (pp. 311-343). Hillsdale, NJ: Lawrence Erlbaum Associates.

Simon, M. A., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91-104.

Smith, E., & Confrey, J. (1994). Multiplicative structures and the development of logarithms: What was lost by the invention of function. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics*. Albany: State University of New York Press.

Smith, E., Haarer, S., & Confrey, J. (1997). Seeking diversity in mathematics education: mathematical modeling in the practice of biologists and mathematicians *Science and Education, 6*, 441-472.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Strom, A. (2008). A case study of a secondary mathematics teacher's understanding of exponential function: An emerging theoretical framework. Unpublished doctoral dissertation. Arizona State University.

Thompson, P. W. (1988). *Quantitative concepts as a foundation for algebraic resoning: Sufficiency, necessity, and cognitive obstacles.* Paper presented at the Annual Meeting of the North American Chapter for the International Group for the Psychology of Mathematics Education, Dekalb, Illinois.

Thompson, P. W. (1990). A theoretical model of quantity-based reasoning in arithmetic and algebra, from http://pat-thompson.net/PDFversions/ATheoryofQuant.pdf

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorm of calculus. *Images of rate and operational understanding of the fundamental theorm of calculus, 26*(2-3), 229-274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.

Thompson, P. W. (2008a). *Conceptual analysis of mathematical ideas: Some spadework at the foundation of mathematics education.* Paper presented at the 32nd Annual Meeting of the International Group for the Psychology of Mathematics Education, Morelia, Mexico.

Thompson, P. W. (2008b). *One approach to a coherent K-12 mathematics. Or, it takes 12 years to learn calculus*. Paper presented at the Pathways to Algebra Conference.

Thompson, P. W., Miller, C., Lage-Ramirez, A., Cruz-Aponte, M., Castillo-Garsow, C., Lima, S., et al. (2008). *Affecting teachers' images of what and whom they teach.* Paper presented at the National Council of Teachers of Mathematics Research pre-session Salt Lake City, Utah.

Thompson, P. W., & Thompson, A. G. (1992). *Images of rate.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, part I: A teacher's struggle. *Journal for Research in Mathematics Education, 25*(3), 279-303.

Trigueros, M. (2008). *Modeling in a dynamical system course.* Paper presented at the Eleventh special interest group of the Mathematical Association of America on research in undergraduate mathematics education.

Verhulst, P.-F. (1977). A note on the law of population growth. In D. Smith & N. Keyfitz (Eds.), *Mathematical demography: Selected papers* (pp. pp. 333-337). Berlin: Springer-Verlag.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Shoenfeld & J. Kaput (Eds.), *Research in collegiate mathematics education IV* (Vol. 8, pp. 103-127). Providence, RI: American Mathematical Society.

APPENDIX A

ORIGINAL TASK PROTOCOLS

Task 1 **OUTLINE**

Jodan Bank uses a "simple interest" policy for their EZ8 investment accounts. The value of an EZ8 account grows at a rate of 8% of the initial investment per year. Create a function that gives the value of an EZ8 account at any time.

GOALS

- Develop a (bi)linear simple interest model to use for later tasks.
- Students see the simple interest function as a function of a parameter (initial investment, and a variable)
- Evaluate students' thinking about time.
  - Do they think of time as a continuously varying quantity (conceptual time)?
- Students will need to think about the growth rate of the account
  - Establish early that the "growth rate" of the account always means dollars per unit time, not 8%.

ANTICIPATED DIFFICULTIES

- Actually constructing the function will be easy for the students. Can't end the interview too early without achieving the other goals.
- Getting students to talk about time in a way that reveals how they are thinking about it, or if they are thinking about it.
- Hypothesis: Students will have difficulty thinking of the value of the investment as a quantity that varies, and prefer to think of the investment as having a specific value at a specific time.
- Students may not make a distinction between the current value of the investment and the initial value of the investment.
- Students will see different accounts with different investment amounts as different situations that must be modeled separately (models-of) rather than as a single parameterized model. Pat: Yes. I think to anticipate this would require that you *let* them model several specific situations and then use a parameter to generalize.

QUESTIONS

- Greetings and small talk
  - New environment means I have to put them at ease. Easy problem means I have time to do so.
  - Reduce small talk over later interviews, and as the tasks become more interconnected.
- Understanding the context
  - Derek: Read the problem out loud
  - Tiffany: Would you explain the situation in your own words?
    - Derek: Anything to add?
  - Derek: What does "8% of the initial investment per year" mean?

        - Want to draw out here the idea that the amount of interest accumulated in a year depends on two variables. Ask follow-ups to either student as needed:
        - So if Phil opens an EZ8 account with $500, what would his growth rate be?
        - If Patricia opens an EZ8 account with $7500, what would her growth rate be?
- In a few minutes, I'm going to have each of you write down your own function (indicate the function described in the question). But first, talk a little bit about what what this function is going to be modeling.
    - Issues of step function vs linear functions
        - What does "at any time" mean?
            - At any day or any time, a customer could check their account balance. The bank needs a policy for how it reports the value of the account at any time. What do you think would be a reasonable policy?
                - For example, if Phil checks his account 3 months in (0.25 years), what might the bank tell him that his account balance is?
            - What the bank actually does is called "prorating." Takes the fraction of the year that has passed and adds the same fraction of the yearly interest to the account. If Phil inquired about his earned interest at the end of a quarter of a year, the bank would tell Phil that he has earned a quarter of the interest he would have earned in a year. The same for 0.10 years, 0.118 years, 0.375 years, 0.428 years, etc.
- Go ahead and write down a function that gives the value of Phil's EZ8 account after any amount of time (in years) since he deposited $7000.00.
    - If they need some help with the parameterization:
        - Try starting with Phil's account.
            - What is the value of the investment when 0 years have passed?
            - What about when 2 years have passed?
            - If a quarter of a year goes by, how much does Phil's account change by?
        - What about Patricia's account? What's different? What's the same?
- Poking around conceptual time
    - Patricia has a very busy schedule, but whenever she has a chance, she checks the value of her account. What will Patricia see?
        - Patricia travels a lot and has very strange hours. Sometimes she can check her account twice a day, sometimes it's weeks before she can check her account value.

- How can she figure out ahead of time the change in her account since the last time she checked?
  - Potential problem: students will try to answer with a number or with a rate (some number per year)
  - Looking for: Size of the changes depends on the amount of time between checks.
  - Tell me about the bank’s prorating policy.

Task 2 **OUTLINE**

The competing Yoi Trust has introduced a modification to Jodan's EZ8, which they call the YR8 account. Like the EZ8 account, the YR8 earns 8% of the initial investment per year. However, four times a year, Yoi Trust recalculates the "initial investment" of the YR8 account to include all the interest that the customer has earned up to that point.

Fred deposited $1250 in a YR8 account. Create a function that gives the value of Fred's YR8 account after any amount of time (measured in years) since he made his deposit (assuming he makes no deposits or withdrawals).

GOALS

- Students see the value of the account as something that grows in continuous time.
- Students observe a geometric growth in the principal
- Students are aware that the rate of change of the investment with respect to time increases every compounding period
- Students see the compound interest function as being constructed from simple interest scenarios.

HYPOTHESES

- Students will have to use a simple interest model to calculate the interest at the end of each compounding period.
- The geometric model will emerge from a pattern of calculated principals where the calculations are left unevaluated (in open form).
- Students will have to alternate between simple interest and geometric scenarios (what happens at the moment of compounding) in order to make sense of compounding periods that are "smaller" than the rate's time unit.
- Students will initially see the GC graph as either a straight line or as a curve, not as a piecewise linear function.

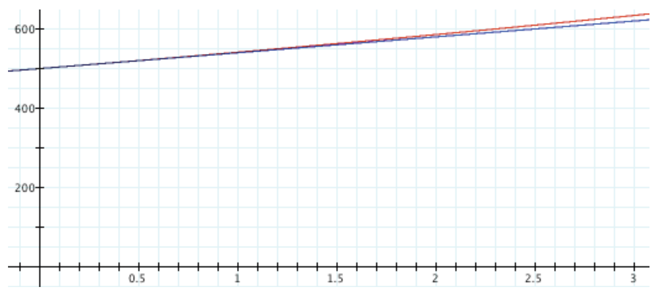
QUESTIONS

- Less small talk
- Understanding the context
    - Tiffany: Read the problem aloud
    - Derek: Would you explain the situation in your own words?
        - Tiffany: What does this "recalculates the 'initial investment' mean?
        - Derek, Explain what Tiffany told me. Do you have anything to add?
    - Imagine that Phil has invested $500 in an EZ8 account, and Patricia has invested $500 in a YR8 account. What happens to each account during the first three months?

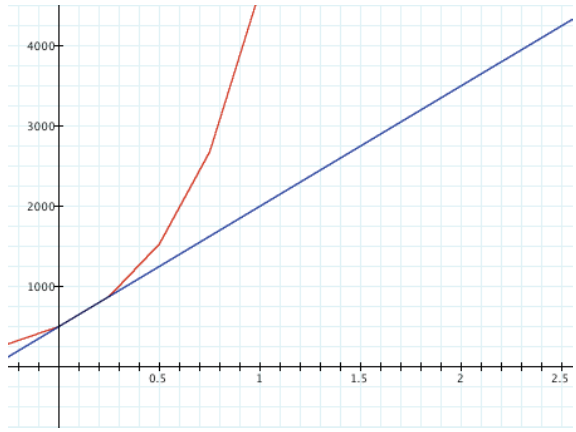
- If Phil were to check his balance at the end of three months, what would he see? What about Patricia?
    - How much interest would Phil earn in a year?
        - How much interest would Phil earn in 1/4th of that time?
        - How much would Patricia earn in that 1/4th of a year?
    - If Phil were to check his balance after one month, what would he see? What about Patricia?
- What happens after six months? How would you calculate the value of Patricia's account at the end of six months?
    - How fast in dollars per year is Patricia's account growing during these three months.
    - Try thinking about it this way. Imagine that after three months, Phil took all the money out of his EZ8 account, receiving his earned interest on $500 for 3 months. Then he invests it all immediately in another EZ8 account, and repeats this every 3 months. Is this like or unlike Patricia's YR8 account?
        - How much would he earn in 1/4th a year?
        - How much would Patricia earn in a 1/4th of a year?
- After nine months?
    - Create a table of # of years (increment in quarters), Value of the Phil's account at that time, and interest Phil earned in that quarter, value of Patricia's account, and the amount of interest Patricia earned in that time.
- What would be the value of Patricia's account at the end of *n* quarters?
- What is the value of Phil's account after 0.6 years?
- What about Patricia's account?
    - Patricia's account was worth $.... after 0.5 years. Between 0.5 and 0.6 years, how much interest did Patricia's account earn?
- Explain the behavior of Phil's account over a whole year.
    - If students a fixed on quarters, ask about specific times in between.
- Explain the behavior of Patricia's account over a whole year.
    - If students a fixed on quarters, ask about specific times in between.

- Terminology time
    - When the bank says that Patricia's account is compounded quarterly, what does that mean?
    - Quarterly means "four time a year." What happens four times year?

        - The value of the account at the beginning of the compounding period is called the principal.
        - What's important about the principal?
            - At what rate does Patricia's account grow between 3 months and 6 months? How would you calculate it?
                - In the event of difference quotient: What if you only know the value of the account at 3 months?
            - The growth rate of the account during that interval is based on the principal.
    - So summarize, what does "compounding" mean?
- Go ahead and agree on a function that answers the question.
- Depending on the difficulties that arise
    - What is the function going to be modeling?
    - Could you write a function for the value of Patricia's account over the first quarter only?
        - [after function is written] When is that function valid?
        - So if that function isn't valid for the second quarter, What would be the function that correctly predicts the value of Patricia's account over the second quarter only?
            - What is the rate of growth in dollars per year of Patricia's account during the second quarter?
                - From this point on "rate of growth" will always mean dollars per year. The interest rate will be called the "per-capita rate"
                - After 0.4 years, how long has Patricia's account been growing at that rate?
            - If students want to write f(.25)*0.08(x) rather than f(1)*0.08(x-0.25)
                - What does x represent?
                - What did we decide the function was modeling?
                - How can we write these functions so that *x* always means the same thing?
            - When is this function valid?
    - Here's how you write a piecewise definition.
    - What is the function for the value of Patricia's account over the 7^{th} year only?
        - What is the rate of growth of Patricia's account during the 7th year?
            - During the nth year?
        - What is the function for the value of Phil's account over the nth year?
        - What is the value of Phil's account after 13.78 years?
- Graphing calculator

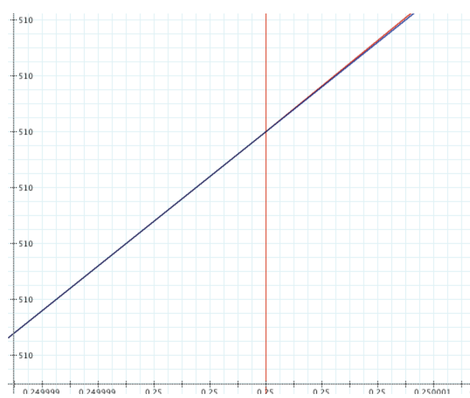
- Enter your functions into graphing calculator. Let's see what they look like.
    - Anticipate: Students will try to graph each line as a separate function, rather than a piecewise function.
        - Cntrl-f and cntrl-a
- Explain to me what the graph is showing
    - If I were to select a point on this graph, what would it mean?
    - What about this point? This one?
    - What if I pick a time? Say 19.37218 years?
    - Is there any time I can pick where the account won't have a value?
        - What about -2?
- Hmm… it looks to me like it's just a straight line. Is it really a line?
    - How could you tell?
    - Try graphing Phil's account value on the same axes, what do you see?
        - 

        - (red Patricia, blue Phil)
    - Now we know Phil's account is a straight line. What about Patricia?
    - What are your functions telling you?
- It's kind of hard to see what's going on. Let's try graphing a similar problem.
    - Imagine that instead of growing by 8%, Patricia's and Phil's accounts both grow by 300%. How would that affect your function definitions? Copy and paste to a new file, so we can come back to our original problem.
    - What does it look like in GC?



-
- What does Patricia's graph look the way it does?
    - How is Patricia's account growing during the first month?
        - What does the bank policy tell you about how Patricia's account grows during the first month?
        - Second month? Third month? Fourth month? Fifth? Sixth? Seventh?

- Back to the 0.08 graph. Is this graph a straight line?
    - How do you know?
    - What do you know must be going on, even if you can't see it very well?
    - Use "shift" clicking to zoom way way in on the neighborhood of 0.25 years. This is where it's the most visible:



    -
    - What do you see? How does Patricia's account behave before 0.25 years? How about after 0.25 years?
    - [See "02g patrica vs phil" for an anticipated final GC product]

- (Optional) Model Critique

- What would happen if after 2.6 years, Phil deposited $1000 in his account?
- Does your function account for a situation like this?
- Why would it be hard to make a function that accounts for deposits and withdrawals?

Task 3 **OUTLINE**

The Savings Company (SayCo) also competes with Yoi Trust and Jodan. SayCo's PD8 account policy is as follows: if you have one dollar in your bank account, you earn interest at a rate of 8 cents per year. For each additional dollar, your interest increases by another 8 cents per year. If you have fractions of a dollar in your account, your interest increases by the same fraction, so 50 cents earns interest at 4 cents per year. Here is SayCo's new feature: At any moment you earn interest, SayCo adds it to your account balance; every time your account balance changes, SayCo pays interest on the new balance and calculates a new growth rate.Why is SayCo's PD8 the most popular account?

Pat(pre-"new feature"): The most popular at SayCo? Or the most popular of PD8, EZ8, YR8? Also, while the PD8 account does address the relationship between aggregate behavior and individual behavior, it doesn't imply continuous compounding, which you seem to say it does in your *anticipated difficulties.* Or, if it does imply continuous compounding, I'm missing it.

GOALS

- Students construct a function that predicts rate of account value with respect to time from the value of the account (essentially differential equation form).
- Establish a student created convention of notation that can be used for all later tasks.
- Compare the differences in the behavior of different accounts over continuous time.
- Students realize that PD8 is a continuous compounding model

ANTICIPATED DIFFICULTIES

- Students will have difficulty extrapolating from the behavior of a single dollar to the behavior of the account as a whole, specifically:
    - The growth of the aggregate is related to the growth of the individual dollar. Growth is distributive. When each % increases by 8 percent of itself, then the whole thing increases by 8% of itself
        - Thinking this way requires recognizing that there is a proportional relationship between amount and interest. 8 cents earned a dollar(an amount) may not mean the same thing as 8% of some principal (a relationship).
    - Students may have difficulty with the nature of continuous compounding. I anticipate that students will want to think of this situation in terms of discrete compounding: After some tiny bit of time passes, the balance updates with earned interest and the growth rate updates accordingly; in contrast to: whenever the amount invested changes, even by fractions of a penny, the balance updates with earned interest and the growth rate updates. Pat: Man! This is a subtle difference!

- Students may have difficulty introducing time into a formulation of growth rate that depends on principal. For any given investment size, the growth rates of the PD8 and YR8 accounts are the same. It is only as the students imagine the value of a PD8 changing that the growth rate increases beyond that of the YR8.
    - But there is no time in the PD8 formulation. Imagining it changing requires the realization that for any increase in time, no matter how small, the value of the PD8 investment increases, and therefore so does the growth rate.
    - This is necessary to compare accounts.
- Average rate of change is traditionally calculated from Δy/Δx, so it depends on two variables. This rate of growth depends only on one variable, account value, and not on time.
- Students may be thinking of the rate of growth as a qualitative thing: fast or slow, measure of how steep, etc, rather than as a number.
- Up until this point, students will have been using generic function notation, such as *y*=…*x*…. or *f*(*x*)=…x…. In that case, *x* means something different in the function created for this task than it did in previous models. Recalibrate notation for consistency.

QUESTIONS

- New account policy for you guys to evaluate.
    - Derek: Read the problem aloud
    - Tiffany: can you explain the PD8 account policy?
        - Derek: What does the interest earned depend on?
            - Note, it depends on both the amount of the investment and the time that has passed.
        - What does the rate of growth of the account depend on?
            - What rate does the account grow at if there is 1 dollar in the account?
            - Two?
            - Two and a half?
            - Does it matter when the account has 2.5 dollars in it?
            - At what rate does the account grow if there is 1 cent in the account?
        - What does the value of the account depend on?
- Construct a function that predicts the growth rate of the account from the value of the account. Pat: Interest rate? Dollars per year?
    - (This is intended to build on the "What does the rate of growth of the account depend on?" question above. Reference that discussion).
    - In this function, What does *y* mean? What does *x* mean?
    - You've written *y* is the rate of change, and *x* is the value of the account. In previous problems [showing EZ8 and YR8 work], *y* was the value of the

account and *x* was time. Let's change some names to make all of these problems consistent

- Earlier you said that the growth rate depended on the account value, and that the account value depends on time, but I don't see that reflected in what you've written. How would you change these functions to reflect the dependencies among the variables?
  - Goal of r(v)=0.08v(t) or r(v(t))=0.08v(t)
- What does the "growth rate" mean?
  - Reintroduce time into the discussion → growth rate is a rate of change w.r.t. time.
  - (Very optional). So the growth rate isn't just any quantity, it's the rate of change of the value of the account (*y*(t), or whatever they call it) with respect to time. Maybe we should call the growth rate variable by a special name that reflects its special relationship to *y*(t).

- Why might the PD8 account be more popular than the other accounts? What kind of reasons would you look for?
  - Which do you think is more popular, the YR8 or the EZ8? Why?
  - Why is PD8 the most popular?
- Imagine that Phil invested $7000 in a YR8 account, and Patricia invested $7000 in a PD8 account.
  - What happens to the growth rate of the YR8 as time changes?
  - What about in the first year?
    - What about in the first month?
    - What about in the first day? Minute? Second?
  - What happens to the value of the PD8 account as time changes?
    - How long will it take value of Patricia's account to change by 1 cent?
      - What is the initial growth rate on Patricia's account?
      - How long would it take to accumulate 1 cent at this rate?
      - What is the value of the value of the account after $1/560^{th}$ of a year?
      - According to your function, what will be the growth rate of the account be at $1/560^{th}$ of a year?
    - What happens to the principal of the YR8 account as time changes?
      - How often is the YR8 compounded?
    - What happens to the principal of the PD8 account as time changes?
      - How often is the PD8 compounded?
  - (optional stuff) What happens to the growth rate of the PD8 account as time changes?
    - (if having trouble, scaffold the composition with account changing -> rate changing)
    - What about in the first year?

        - Month? Day? Minute? Second?
    - (optional stuff) How is the way that the growth rate of the PD8 changes different from the way that the growth rate of the YR8 changes?
        - How will that affect the value of the PD8 account compared to the value of the YR8 account?

Task 4 **OUTLINE**

Previously: *Yoi Trust one-upped Jodan's EZ8 account, which was a simple interest account, with their YR8 account, by adding earned interest to the account value at the end of every 3 months (4 times per year).*

For the Yoi YR8 account, create a function that gives the rate of change of the value of the account (in dollars per year) for every value of the account.

Pat: Again, be specific as to whether you are asking about interest rate (rate relative to principal) or dollars per year (rate of change of account value with respect to time).

GOALS

- Use parametric function reasoning to construct a graph of the growth rate of the account in relation to the value of the account
- Understand how to interpret the graph in such a way that it gives information about time
- Students make predictions about the family of graphs that is parameterized by the compounding interval
    - One graph at a time.
- Conclude that in "continuous compounding" the growth rate of the account is proportional to the value of the account.
    - Hey! That's exactly like the PD8!

HYPOTHESES

- In order to come to an understanding of continuous compounding in this way, the students will need to juggle lots of quantities and relationships at once specifically: growth rate of the account, the value of the account, and time.
- In order to predict how smaller compounding intervals affect the graph, students will have to come to see the compounding interval in the length of the step.

QUESTIONS

- Let's go over the work that we've done on the YR8 account so far
    - Function definition
        - Tell me what this function means.
            - Look for meaning of the function definition in terms of the meanings of the variables (the quantities that they reference)
            - Explain how the function connects to the YR8 policy
    - Function graph (saved from GC work)
        - What is this graph telling you?

- Look for: Meaning of the function graph in terms of the quantities.
- When I click on a point, what information does it give me?
- From 0 to 0.25 years, what is the growth rate of the account? How could I calculate it?
- How could I think about the growth rate of the account at exactly 0.2 years?
    - Does the growth rate every change between 0.19 and 0.21 years?
- How could I think about the growth rate at exactly 0.25 years?
    - What is the growth rate between 0.24 to 0.25 years?
    - What is the growth rate between 0.25 and 0.26 years?

- Go ahead and make the graph of the growth rate and the value of the account
    - Use the graph of account value and time, and what you know about finding the rates of growth.
    - When the investment begins, what is the account value?
        - What is the growth rate?
    - At 0.1 years, what is the value of the account?
        - What was the growth rate?
    - What happens to the value of the growth rate between 0 and 0.1 years?
    - What happens to the value of the account between 0 and 0.1 years?
    - What about 0.2 years?
    - 0.25 years?
    - 0.3 years?
- Interpreting the graph.
    - What is the graph showing you?
    - What is happening at these “jumps” or breaks?
    - When are these happening? How many years until the first “jump?”
    - When I calculate a new growth rate at one of these jumps, what does that growth rate have to do with the value of the account?
- Limiting compound interest?
    - What would the graph look like if I compounded 5 times a year instead of 4 times a year?
    - At the end of the first compounding period, what would the value of the account be?
        - How could I calculate the new rate of change using that information?
        - At the end of the second compounding interval, how much would the value of the account be compared to when I compound 4 times a year?

- You can be very approximate.
- If I knew that value, how would I calculate the rate of change?

- Sketch a graph of what you think the graph would look like if the article compounded 8 times a year.
  - Show them on GC
- If I compounded 9 times a year?
- 10 times?
- 20 times?
- Every day (365 times?)
- Every minute?
- Every second?

- Write a function that would calculate an approximate rate of change from the value of the account if I compounded every second.
  - How accurate is it?
  - Have we seen this relationship before?
- The PD8 account policy was written very differently, but we wound up with the same relationship. What gives?
  - What happens to the PD8 account value as time changes just a little bit? Just a fraction of a second?
  - What happens the rate of change of the PD8 account whenever the PD8 account value changes?

Task 5 **OUTLINE**

[Adjust variables to match students' pre-existing notation, if it exists]

--- may just ask the task questions verbally ---

Create a graph showing the relationship between the growth rate of the value of a Jodan PD8 account and the value of the PD8 account.

Use this information to construct a graph showing the relationship between the value of the PD8 account and time.

GOALS

- Students understand phase plane as a parametric curve that is dependent on time.
- Students approximate a graph of the solution using compound interest reasoning
- Students compare the rates of change across successive compounding intervals, and make conclusions about the amounts of change over each interval for successive intervals
    - The next interval has a larger rate of change
    - The account value over the next interval will change more than the previous interval

HYPOTHESES

- Students will not initially see the phase plane as giving any information about how the value of the account behaves in time, because time is not a quantity in the phase plane graph.
    - In other words, students will not look at the phase plane graph and "see" time.
- In the previous task the students actually saw what the graph of a PD8 vs. time looked like (at the end of the GC demo). Students may simply recall that graph and sketch it without really thinking about it.
- Students will initially want to think about time in big chunks… ex: what is the account value after 1 year, after 2 years.
- Students will get "hung up" on everything changing all at once.
    - In order to succeed in the task, a student will need to
        - Fix a small interval of time.
        - Choose a starting principal for that interval
        - Find the rate of change for that principal
        - Imagine that the rate of change is fixed for the entire small interval of time
        - Find the ending principal for that interval
- Students will not understand the question as asking for an approximate graph
    - Students will believe that fixing the rate of change over a small interval will have a bigger error impact than it actually does

- Students will want to calculate numbers, rather than reasoning about the qualitative behavior of relationships between successive intervals.

QUESTIONS

- small talk, just a tad
- Do you remember this function for the PD8 account?
  - What is this function telling you?
  - What does *y* represent? *x*?
  - How does this function relate to the PD8 policy described in task 3? (Show them).
- Create a graph showing the relationship between the growth rate of the value of a Jodan PD8 account and the value of the PD8 account.
  - I don't actually anticipate that the students will have difficulty with the quantities here, in classes, they've been pretty used to the axes being just about anything, or they should be by now.
- What is this graph telling us?
  - As the value of the account changes, what happens to the growth rate of the account?
  - As time increases, what do you expect would happen to the principal?
    - At this point, I'm happy with just "it increases"
    - Anticipate that students might interpret this as a question about what the graph.
      - Close your eyes and just forget forget about the graph for a minute. You're Phil. You've just put some money in a new account. The next day, what do you expect to happen to the value of the account? What about the day after that? After that?
  - As the value of the account increases, how quickly does it increase?
- Use this graph to construct a graph showing the relationship between the value of the PD8 account and time.
  - If students sketch the graph from memory, insist on "at least a reasonably accurate approximation."
  - If students get hung up on big chunks. Ask them to think about what happens on the first day of the investment.
  - If students get hung up on everything changing all at once
    - Ask them to think about one quantity at a time
      - Sketch out your axes for the graph that you're going to make
        - What is measured on these axes?
      - Let's start with the beginning of the first day. What do we know/need to know about the investment?
        - How much was invested.

            - Ok, we know how much was invested. What else do we know, from the information that we have?
                - Current time is blah, current rate of growth is blah blah.
            - Tick time forward a day. About how much did the investment grow?
                - How much of a year is a day?
                - About how fast is the investment growing during this day?
            - What do we know about the account at time?
    - If students object to fixing the rate of change (unlikely?)
        - I will jump up and down for joy.
        - What does it mean if we fix a rate of change for a period of time, and then update it every once in a while?
            - Imagine that we fixed it for a year, and then updated the rate at the end of every year.
        - Reference, but do not show the GC compound interest limiting demo
            - What would our YR8 graph look like? If I were using the GC demo, how would I make a graph of the YR8 vs time?
            - How would I make the YR8 graph look more like the "real" PD8 graph?
    - If, after a few small intervals of time, the students continue to get hung up on numbers
        - Let's stop with the numbers
        - What would the next account value be like, compared to this current account value?
        - What would the next rate of change be like compared to this rate of change?
        - So how much interest do you expect to earn today, as compared to yesterday?
            - What about the day after?
            - What about the day after?

Task 6 **OUTLINE**

[PD8 model here. Use students' own notation.]

At the turn of the 19th century, Thomas Malthus proposed this financial model as a model for predicting the population of the world. Using the properties and the behavior of the model, describe the good and bad points of using it as a population model.

GOALS

- Students consider a mathematical expression as a description of a relationship that can be applied to model different situations (model-for)
- Students consider a model to be an object of critique
- Students understand that the Malthus model is a poor model for both very large (unlimited growth) populations, and very small populations (overly smoothed)

HYPOTHESES

- Students will initially have difficulty thinking of population as a quantity that has a rate of change.
    - Students will initially think of population in terms of individual experience. A family has a child, a few years later has another child, Then after that stops having children.
        - A rate of change in this situation is impossible
- Student will need to think in terms of large populations and aggregate behavior for it to make sense that a population has a rate of change.
- On a small scale, students might think of "how fast" a population is growing in terms of a per-capita or per-family rate, rather than as an average rate of change.
    - Difference between "this family has more kids in the same time period" and "this family has kids more frequently"

QUESTIONS

- Opening with some "small talk."
    - So tell me, do you have any bothers or sisters?
    - How many? How old are they?
    - Let's create a graph that shows the population of your family over time.
        - Let's start time at when your parents got married.
        - If you don't know exactly, it's ok to fudge a little bit. Say they married 2 years before the first kid.
        - What is the population at 1.1 years? At 2.73 years? At 14.1745 years?
    - What would happen to the world if everybody was like your parents?
        - Imagine what your neighborhood would be like if every couple had the same number of kids as your parents did.

- What would a graph of two families look like? Say, yours and your neighbors?
  - Three families?
- Are all the kids the same age?
- Do they all stay in the same neighborhood?
- What happens to the parents?
- Do the kids have kids?
  - What would a whole city of your parents be like?
  - What about the whole world?
- You know, my mom's parents had 7 kids.
  - What do you suppose the graph of her family's population might look like?
  - What about three families like hers?
    - Is there any reason to think that the parents would start having kids at the same time?
  - What would happen to the world if everybody got married and had seven kids?
- Let's look at some of the graph's we've created. Say the "three of your families" and the "three of my grandparents' families" graphs.
  - How could we think about how fast the family is growing?
  - What about "two of my grandparents" vs "one of my grandparents"
  - What about a city of 10 million people?
    - Who act like your parents?
    - Who act like my grandparents?

- Present the question
  - Read the question out loud
    - What is the question asking?
  - Going back to the initial bank policy (show problem), what does the 0.08 mean?
    - What would it mean if we were talking about people?
    - What would be a reasonable per-capita rate for a person?
      - What are some differenced between the way a dollar earns interest and the way a person produces offspring?
        - Do the 8 cents come all at once?
        - What about children?
        - What does your model say should happen?
          - How many children would the parent have after a half of a year?
        - What if there were 100 million parents? Would their children come all at once?
          - Think about some of the family graphs we made. What happens when there are more parents?

- What does the PD8 model say about the relationship between the population and the rate of growth of the population?
  - Can you give me an example?
  - Does this make sense?
- Going back to our previous work, What does the growth of a population look like over time? Is this the same sort of behavior that you predicted in the case where everybody is like my grandparents?
- What does the model predict if there are 0 people in the population?
- What does the model predict if there is only one person in the population?
  - Whoa whoa whoa, how can 1 person have kids? Maybe we need to talk about amoebas, or maybe the model measures couples, or maybe we only count women.
- What if there are 10? What would the first year be like?
- What if the population started with 10 million? (A large city)
- What if the population started with 100 billion? (14 times the population of the earth)

Task 7 **OUTLINE**

[Student notation Malthus model here, in per-capita rate of change form if this has been developed.]

Verhulst proposed the following modification to the Malthus model in order to make it more realistic. Instead of imagining that the per-capita rate of change was always constant, what would happen if that rate slowed down in response to population pressure, eventually becoming 0 for some population value called the carrying capacity. How would you write the model?

GOALS

- Students construct a per-capita rate of change / population graph for the Malthus model
  - Students interpret this graph as being about the behavior of every single organism, based on the total population
    - Need to ask the question: why would the size of the population affect what one person does?
- Student construct a per-capita rate of change / population graph of the Verhulst model
  - Students can describe the biological mechanism for slowing birth rate.
    - Note, this doesn't cover why the slowing birth rate is linear, that's ad-hoc. Any decreasing curve will be acceptable here.
- Students construct a per-capita rate of change / population function for the Verhulst model

HYPOTHESES

- This is so far in the future that all my hypotheses will change.
- Students will not see a reason to graph per-capita rate of change / population, since per-capita rate of change is not a function of population.
- Students will not be thinking about overpopulation problems, or availability of resources.
  - These will need to be introduced in order for students to make sense of the problem with the Malthus model that Verhulst tried to fix
- Students will not see per-capita rate of change as arising from a process of division, but will instead intuit the per-capita rate of change from imagining the situation.
  - Need a notation for per-capita rate of change in order to talk about it.
  - Need to talk about the relationship between per-capita rate of change and the rate of growth.
- If students really think covariationally (Ha! I should be so lucky), the Verhulst model will present a problem
  - There is no reason why the per-capita rate of change should change linearly with population. Thinking about how per-capita rate of change

changes as population changes will only bring out that this linear assumption doesn't make any sense.

QUESTIONS

- Ok, so here is our Malthus model. Previously we talked about some of the ways that the model is unrealistic. Tell me about them.
    - What about when the model has a very high population?
    - What should happen when the model has a very high population?
        - Why would this happen? Why can't the population grow forever?
        - What would happen to each organism when the model has a high population?
            - Why would the size of the population affect what one person does?
            - How would it affect their (per-capita) birth rate?
- Let's make a graph of how population affects individual behavior in the Malthus model
    - What two things would we be graphing?
    - Make the graph
    - What does the Malthus model say about individual behavior at high populations?
- Let's make a sketch of a graph of something that might be more realistic. What should happen to the per-capita rate of change as the population gets bigger?
- Ok, Now a guy named Verhulst made a suggestion for how to fix the problem. Here's his suggestion
    - [Show the question]
    - Tell me a little bit about what Verhulst is proposing.
    - What is the carrying capacity?
    - Sketch a graph of what Verhult is proposing. What is he saying about how population would affect per-capita rate of change?
        - How can I interpret this graph?
        - What does it tell me about how organisms behave?
- Write a function definition for this graph.
- If I know the per-capita rate of change, and I know the population, how can I figure out the rate of change?
    - For the Malthus model?
    - For the Verhulst model?
- Write a function for the Verhulst model that gives the rate of change of the population based on the population.

Task 8 **OUTLINE**

[Adjust variables to match students' pre-existing notation, if it exists]

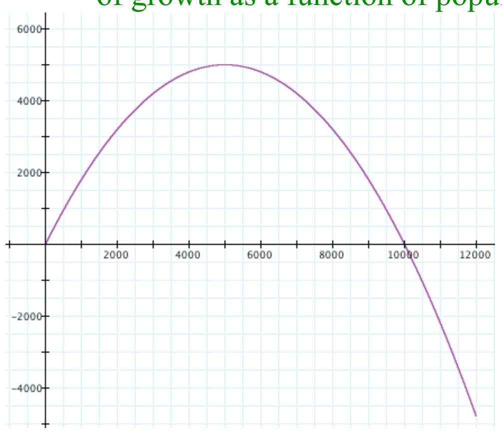
Last time, we developed this model for population growth:
[Verhulst model in student notation; note: graph is r=2, K=10,000. Adjust parameters as needed]

Create a graph showing the relationship between the rate of change of the population with respect to time, and the population at that time.

Use the graph you created to construct a graph showing the relationship the population and time.

GOALS

- Students construct a graph of Verhulst model phase space.
    - Students interpret that graph as a parametric graph
    - Alternatively, students interpret population as a function of time, and rate of growth as a function of population.



- Student make a connection between their approach to the PD8 time series (task 5) and creating the Verhulst time series.
- Students compare the rates of change across successive intervals, and make conclusions about the amounts of change over each interval for successive intervals
    - The next interval has a larger rate of change
        - the population over the next interval will change more than the previous interval
    - The next interval has a smaller rate of change
        - The population over the next interval will change less that the previous interval
- Students understand that if, over any time interval, the rate of change of the population passes through zero, that's it. Game over, man.

HYPOTHESES

- Students will interpret intervals as having a beginning and an end, but not think about the behavior in the middle.
    - This will result in the students predicting oscillatory behavior around the carrying capacity
- Students may believe that in the region where the graph is positive, the function is always increasing at an increasing rate
- Students will repeat the struggles they had with the PD8 phase plane, whatever those turn out to be.

QUESTIONS

- Last time, you came up with this model for describing the way in which a growing population behaves. What is this function telling us?
- Create a graph of this function (can use GC, cause I don't really care).
    - What information does this graph tell us about how the population behaves?
        - What is being measured on each axis?
        - As the population changes, what happens to the rate of growth?
        - How does the rate of growth affect the population?
        - How does time into all this?
            - How can population change without time?
            - Ok, so time changes, what happens to the population?
                - How fast?
            - Population changes, what happens to the rate of change?
            - Rate of change changes, what happens to the population?
                - What does "it changes faster" or "it changes slower" mean? Tell me what's going on in terms of population and time.
- Let's start working on what a population(time) graph would look like?
    - Where do we start?
        - We need a starting time and a starting population.
        - Then what happens?
            - What happens with time?
            - Let's look over a small piece of time. What happens to the population as time changes? How much do you estimate that the population would change by?
                - What does the rate of change have to do with how much the population changes?
                - What does time have to do with how much the population changes?
        - Filling in the middle

            - What happens to the rate of change as the population changes from (blah) to (blah blah)
            - What happens to the rate of change of the population as time changes over this interval
                - How can that be reflected in your graph?

- Go ahead and make your graph of population(time)
    - Talk to me about what's happening when the population gets close to the carrying capacity.
        - What happens to the rate of change as the population changes from (blah) to (blah blah)
        - What happens to the rate of change of the population as time changes over this interval
            - How can that be reflected in your graph?
        - What would the graph looked like if we used smaller pieces of time?
        - What if the population started at the carrying capacity? What would happen then?
        - Could the population ever pass through the carrying capacity? Could it ever start lower and then go higher?
            - What would have to happen to the popoulation in order for that to occur?
                - Hold out for "it would have to pass through K"
            - What would have to happen to the rate of change, in order for that to occur?
                - Hold out for "it would have to pass through 0"
                - If the rate of change were 0, for any moment. Could the population keep changing?
                - What would happen to the rate of change if the population always stayed the same
                    - Either, it would always be 0, or it would always stay the same, or the population staying the same means that the rate is 0 are all good.
        - Issues of asymptotic behavior.
            - If the smaller and smaller intervals doesn't get them there, I'm just going to give them this tidbit. They can take that on faith until they get to a differential equations course.

APPENDIX B

SAMPLE ASSENT FORMS

ARIZONA STATE UNIVERSITY
Student Assent Form

[Date]

Dear [FirstName],

I am Dr. Pat Thompson, Professor of Mathematics Education in the Department of Mathematics and Statistics at Arizona State University. This year I will be working with [Teacher]'s Math [Level] class as part of a research project to investigate ways to improve high school students' mathematical learning.

My graduate students and I will be in your mathematics classroom while [Teacher] teaches. We will help her develop central ideas of mathematics with the aid of computer software used widely by practicing mathematicians and engineers. I will study what students actually learn from this approach and to study difficulties they might have with it. To study what students learn and the difficulties they have we must videotape classroom instruction and interviews with students. My graduate students and I can then study the videotapes as part of our effort to determine what students "have in mind" as they think about the ideas they've been taught, which then informs our design of future instruction.

I send this letter because I wish to include you in videotapes of classroom instruction. Also, I hope to interview you periodically and to videotape those interviews. The interviews will be conducted at school, during the school day, and last approximately 20 minutes. You will be paid $10 per interview. Interview questions will focus on what you understand about the mathematical ideas taught during class. For example, if the current topic is average rate of change, we might ask about situations that probe how you think about relationships between time and distance. *These interviews are not tests. Your grade will not be affected negatively in any way by participating in them.*

We will also collect background information on students in [Teacher]'s class so that we can compare her class with national averages. This will include standardized test

Tempe CAMPUS
CRESMET
Center for Research on Education in Science, Mathematics, Engineering, & Technology
BOX 873604, TEMPE, ARIZONA 85287-3604
VOICE: (480) 727-8884 FAX: (480) 965-5993 E-MAIL: CRESMET@ASU.EDU

data collected will remain entirely confidential and will be stored so that it cannot be associated with individual names. Videotapes will be stored in a secure location at scores, gender, grades, and ethnicity. None of this data will be linked to your name. All Arizona State University and digital copies will be stored on a secure server. We might show segments of videotapes during presentations of our research results at professional conferences, and we might show segments in courses on methods of teaching mathematics. Transcribed excerpts from class instruction or individual interviews might be included in published reports of the project. Students in them will be depicted anonymously. Similarly, while data about students' class performance or academic background might be reported in research publications, all students will remain anonymous.

If you wish to not appear in videotapes of instruction, then you will be seated out of camera range to minimize your chance of appearing on tape, and any segments that have you in them will not be shown publicly.

I would appreciate your signing this letter and returning it to [Teacher] by [DueDate]. A copy will be returned to you. Please note that you may withdraw your assent to be videotaped or interviewed, or withdraw from the project, at any time without penalty.

Please call me at (480) 965-2891 (office) or 480-277-1684 (home) if you have any questions. Please call ASU's Institutional Review Board at (480) 965-2179 if you have any concerns about this project.

Sincerely yours,

Pat Thompson

Patrick W. Thompson, Ed. D.

☐ I agree to be videotaped during classroom instruction.

☐ I agree to participate in interviews about the mathematical ideas taught in class, and to be videotaped during those interviews.

☐ I want to stay in [Teacher]'s class, but to not be interviewed or to be videotaped during instruction.

☐ I want to be transferred to a different class.

______________________________ ____________________

[FirstName] [LastName] *Date*

ARIZONA STATE UNIVERSITY

Parent Permission Form

[Date]

Dear Parent or Guardian of [FirstName] [LastName],

I am Dr. Pat Thompson, Professor of Mathematics Education in the Department of Mathematics and Statistics at Arizona State University. This year I will be working with [Teacher]'s Math [Level] class as part of a research project to investigate ways to improve high school students' mathematical learning.

My graduate students and I will be in [FirstName]'s mathematics classroom while [Teacher] teaches. We will help her develop central ideas of mathematics with the aid of computer software used widely by practicing mathematicians and engineers. I will study what students actually learn from this approach and will study difficulties they might have with it. To study what students learn and the difficulties they have we must videotape classroom instruction and interviews with students. My graduate students and I can then study the videotapes as part of our effort to determine what students "have in mind" as they think about the ideas they've been taught, which then informs our design of future instruction.

I send this letter because I wish to include [FirstName] in videotapes of classroom instruction. Also, I hope to interview [FirstName] periodically and to videotape those interviews. The interviews will be conducted at school, during the school day, lasting approximately 20 minutes. [FirstName] will be paid $10 per interview. Interview questions will focus on what [FirstName] understood of the mathematical ideas taught during class. For example, if the current topic is average rate of change, we might ask about situations that probe how [FirstName] thinks about relationships between time and distance. *These interviews are not tests. [FirstName]*'s *grade will not be affected negatively in any way by participating in them.*

Tempe CAMPUS

CRESMET

Center for Research on Education in Science, Mathematics, Engineering, & Technology

BOX 873604, TEMPE, ARIZONA 85287-3604

VOICE: (480) 727-8884 FAX: (480) 965-5993 E-MAIL: CRESMET@ASU.EDU

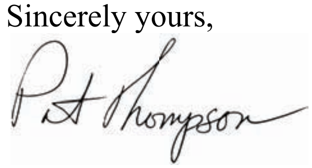
We will also collect background information on students in [Teacher]'s class so that we can compare her class with national averages. This will include standardized test scores, gender, grades, and ethnicity. None of this data will be linked to individual students' names. All data collected will remain entirely confidential and will be stored so that it cannot be associated with individual names. Videotapes will be stored in a secure location at Arizona State University and digital copies will be stored on a secure server. We might show segments of videotapes during presentations of our research results at professional conferences, and we might show segments in courses on methods of teaching mathematics. Transcribed excerpts from class instruction or individual interviews might be included in published reports of the project. Students in them will be depicted anonymously. Similarly, while data about students' class performance or academic background might be reported in research publications, all students will remain anonymous.

Students wishing not to appear in videotapes of instruction will be seated out of camera range to minimize their chance of appearing on tape. Any segments having these students in them will not be shown publicly.

I would appreciate your signing this letter and having [FirstName] return it to [Teacher] by [DueDate]. A copy will be returned to you. Please note that you or [Firstname] may withdraw consent to be videotaped or interviewed, or withdraw from the project, at any time without penalty.

Please call me at (480) 965-2891 (office) or 480-277-1684 (home) if you have any questions. Please call ASU's Institutional Review Board at (480) 965-2179 if you have any concerns about this project.

Sincerely yours,



Patrick W. Thompson, Ed. D.

- ☐ I give permission for my child, [FirstName] [LastName], to be videotaped during classroom instruction.
- ☐ I give permission for my child, [FirstName] [LastName], to participate in interviews about the mathematical ideas taught in class, and to be videotaped during those interviews.
- ☐ I want [FirstName] to stay in [Teacher]'s class, but to not be interviewed or to be videotaped during instruction.
- ☐ I want [Firstname] to be transferred to a different class.

______________________________ ____________________

Parent or Guardian of [FirstName] [LastName] *Date*

APPENDIX C

HUMAN SUBJECTS EXEMPTION

From: "Susan Metosky" <Susan.Metosky@asu.edu>
Date: August 23, 2007 1:32:57 PM GMT-07:00
To: "Pat Thompson" <Pat.Thompson@asu.edu>
Cc: "Debra Murphy" <Debra.Murphy@asu.edu>
Subject: **The Effect of Teaching**

Dear Patrick Thompson,
The IRB reviewed the continuation of your project "The Effect of Teaching for Meaning with the Support of Technology." The IRB determined that this project meets criteria for exemption under Federal Regulations 45CFR46.101(b)(1):http://www.hhs.gov/ohrp/humansubjects/guidance/45cfr46.htm#46.101
Research conducted in established or commonly accepted educational settings, involving normal educational practices, such as:

i. research on regular and special educational instructional strategies, or

ii. research on the effectiveness of or the comparison among instructional techniques, curricula, or classroom management methods.

Research may continue on this project.
Susan

Susan Metosky, MPH IRB Administrator
Research Compliance Office
Admin B Room 371
Arizona State University
Tempe AZ 85287-1103 (Mail Code 1103)
Telephone: 480 727-0871 Fax: 480 965-7772
Susan.Metosky@asu.edu
http://researchadmin.asu.edu/compliance/irb/

From: "Alice Garnett" <Alice.Garnett@asu.edu>
Date: April 27, 2009 10:41:55 AM MST
To: "Pat Thompson" <Pat.Thompson@asu.edu>
Cc: "Scott Courtney" <Scott.Courtney@asu.edu>, "Chris Miller" <cdmille1@asu.edu>, "Sharon Lima" <Sharon.Lima@asu.edu>, "Ana Lage Ramirez" <Ana.Lageramirez@asu.edu>, "Carlos Castillo-Garsow" <cwcg@asu.edu>
Subject: RE: Request to add study personnel

Dear Dr. Thompson,

I have added the study personnel listed below on your studies:
“Developing a Professional Learning Community Model for Secondary Precalculus teachers: A Model for Teacher Professional Growth”, and
“The Effects of Teaching for Meaning with the Support of Technology”
Regards,
Alice

Alice Garnett
IRB Coordinator
Office of Research Integrity and Assurance
Interdisciplinary Building B, Room 371
Arizona State University
(480) 727-6526 phone
(480) 965-7772 fax

alice.garnett@asu.edu
http://researchintegrity.asu.edu/

From: Pat Thompson [mailto:pat.thompson@asu.edu]
Sent: Friday, April 24, 2009 7:25 AM
To: Alice Garnett
Cc: Scott Courtney; Chris Miller; Sharon Lima; Ana Lage Ramirez; Carlos Castillo-Garsow
Subject: Request to add study personnel

Dear Alice,

I would appreciate having Scott Courtney, Chris Miller, Sharon Lima, Ana Lageramirez, Sharon Lima, and Carlos Castillo-Garsow added to the list of study personnel on both of the IRB approvals related to my NSF grant. They are IRB#0607000979 and IRB#0607000988. I have attached their IRB certificates.

Thank you very much,

Pat Thompson