Developing a Mental Image:
The case of one student's struggle with quantification and covariation

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Note: This paper is partially a group effort and partially an individual effort. Portions of this paper were written by Stacey Bowling, Sharon Lima, Kevin Moore, Diana Postelnicu, and Marilyn Carlson, however, none of these authors have reviewed the final product. Portions that these authors have contributed to are marked in the section titles.

Introduction (Bowling, Carlson, & Castillo-Garsow)

It is well documented that even high performing precalculus and calculus students have weak understandings of the function concept and weak covariational reasoning abilities (Carlson, 1998; Thompson, 1994a-c). Studies have revealed that the ability to reason covariationally (that is, to think about formula and graphs as representations of two quantities, and to attend to the varying magnitude of these quantities as they change in tandem) is critical for understanding functions and central concepts of calculus (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Thompson, 1994a) and differential equations (Rasmussen, 2000). For many students, a college algebra course will be their last exposure to these foundational concepts prior to a calculus course; for this reason, we have chosen to focus on student behaviors and abilities in a college algebra course.

Drawing from this literature, we have developed curriculum and instructional supports for college algebra that take a covariational approach to teach ideas of variable, rate of change, function, function composition, function inverse, and exponential growth. Homework assignments and in class instruction also emphasized meaningful communication about functions as representations of covarying quantities.

The purpose of this study is to investigate the development of the college algebra students' understanding of the concept 'function' and their covariational reasoning abilities as a result of their participation in this college algebra course by addressing their interpretation of modeling situations and function representation.

We also conjectured that these students' problem solving abilities as described by the multi-dimensional problem-solving framework (Carlson & Bloom 2005) may be affecting students' learning in the course. As a result our design research study also inquired into the problem solving abilities of students and how their approaches to solving problems interacts with their emerging covariational reasoning abilities and function understandings. Our investigation of the students' problem solving abilities examined how they orient themselves to a problem and how they go about making meaning of problem statements.

This paper represents a small portion of a larger study, in which I investigate one college algebra student's difficulties in orienting to a problem and the impact of those difficulties on her ability to reason covariationally. I investigate her difficulties from the point of view of quantification of a mental image of the problem situation.

In the sections that follow, we begin by describing the background and theoretical perspectives that frame our study. We then describe in detail the intervention and our methods for collecting and analyzing data.

Theoretical Background (Castillo-Garsow & Moore)

The proposed study discussed here is guided by two separate but connected (at least in the college algebra setting) topics: students' covariational reasoning abilities and students' problem solving abilities. I will deal with each of these in kind.

As described above, Covariational reasoning is defined as the "cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Saldanha and Thompson describe understanding covariation as "holding in mind a sustained image of two quantities' values (magnitudes) simultaneously" (Saldanha & Thompson, 1998). In addition to this, these images of covariation are considered developmental. In other words, one first coordinates two quantities' values (e.g., think of the first quantity, and then the other, think of the first quantity, and then the other, etc.). Then, as a student's image of covariation develops, her/his understanding of covariation begins to involve understanding time as a continuous quantity. Thus, the ability to imagine continuous changing quantities begins to form (e.g., as one quantity changes, the other quantity also changes simultaneously). This parallels Dubinski and Harel's (1992) *process conception* of function in that the student is able to imagine the simultaneous changes without having to determine the value of one quantity, and then the value of the other quantity. The two levels of development discussed also exemplify pointwise and across-time views of a functional relationship. In a pointwise view, a student can only analyze a graph point by point, or when presented with a function f(x), they see this as a call to evaluate a point (e.g., an action conception). Contrary to this is the idea of across-time, where the student is able to reason about a graph dynamically. Thus, this allows continuous movement across the independent variable while tracking continuous changes in the dependent variable. Or, in relation to being presented with a function f(x), the student sees this as a general mapping (or relationship) between quantities that can be evaluated for any point within the domain (e.g., a process conception).

. Covariation is at the heart of the mathematical concept of rate of change. As mentioned, understanding rate of change is required for developing an understanding of the major concepts of calculus and differential equations. According to Thompson, a mature image of rate involves i) the construction of an image of change in some quantity, ii) the coordination of images of two quantities, iii) and the formation of an image of the simultaneous covariation of two quantities (Thompson, 1994b). These follow Piaget's three-stage theory relative to children's mental operations about functional thinking with respect to covariation (Piaget, Grize, Szeminska, & Bang, 1977).

Carlson et al. investigated the complexity of students' images of covariation. Namely, the "construction of mental processes involving the rate of change as it continuously changes in a functional relationship" was investigated (2002). High-performing 2nd-semester calculus students were investigated in order to determine their ability to reason about covarying quantities in dynamic situations. It was determined that they were able to construct images of a function's dependent variable changing in tandem with he independent variable but could not construct images of continuously changing rates of change (e.g., the interpretation of inflection points or increasing and decreasing rate). During this investigation, a theoretical framework was created and refined. Initially, multiple behaviors had been identified in undergraduate students involved in interpreting and representing dynamic function situations (Carlson, 1998). In order to classify the behaviors exhibited, a framework that consists of five mental actions and the behaviors associated to these actions was developed. However, this table was not adequate enough to describe a student's covariational reasoning ability. In order to analyze a student's covariational reasoning ability, a collection of behaviors and mental actions must be analyzed. In order to analyze a collection of behaviors and mental actions, the covariation framework was extended to describe multiple levels of covariational reasoning. Thus, the Covariation Framework consists of five distinct developmental levels that consist of the five mental actions (Carlson et al 2002):

Mental Action 1 (MA1): Coordinating the value of one variable with changes in the other

Mental Action 2 (MA2): Coordinating the direction of change of one variable with changes in the other variable

Mental Action 3 (MA3): Coordinating the amount of change of one variable with changes in the other variable.

Mental Action 4 (MA4): Coordinating the average rate of change of the function with uniform increments of change in the input variable

Mental Action 5 (MA5): Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.

In the analysis of a student's covariational reasoning abilities, Carlson et al. note an important observation: students often exhibit behaviors that appear to reveal a high level of development (e.g., L5), but when these behaviors are probed, a student is not able to justify or support the reasoning they appeared to display. This occurrence can be described as a student exhibiting pseudo-analytical behavior (Vinner, 1997). Pseudo-analytical behavior is the situation in which a student does not have the understandings required to meaningfully describe the behavior in which they acted.

Thus, we define the mental action that produced the behavior as a pseudo-analytical mental action. As mentioned, a student is classified as have a specific covariational reasoning ability level only if he or she is able to perform the mental action relative to that level and all levels below. Thus, if a student reveals MA4, they are only classified as L4 if they also exhibit MA1-MA3. If they are not able to exhibit MA1-MA3, the MA4 behavior is a pseudo-analytical mental action.

For this study we will be focusing on a student's ability to move among Mental Actions 1, 2 and 3, and thus the level of covariational reasoning developed. In order to characterize the student's development relative to the class in which they are enrolled, this investigation will involve examining the interaction between students' covariational reasoning abilities and their understanding of central concepts of college algebra (e.g., variable, rate of change, function, function composition, function inverse, and exponential growth). Furthermore, due to the setting in which this framework was developed (e.g., high performing calculus students), the framework will remain open to modification. In fact, it is conjectured that due to the population being analyzed, modification will occur.

Covariation involves coordinating the change of two quantities, so it is necessary to expand on the meaning of quantity used in this paper. One meaning of quantity which provides a useful theoretical perspective is that of Thompson (1994c), where he describes his meaning of quantity as being a result of a measurement. This meaning provides an additional perspective to understanding the orientation phase of problem solving by looking at the process of quantification as an imagined measurement. In order to quantify a problem situation, a person must form a mental image of the physical situation described in the problem, and imagine the act of measuring it.

Following Thompson's meaning of quantity, my use of *a quantity* in this paper will refer to a numerical result of a (real or imagined) measurement. This measurement may not necessarily be obvious, as concepts of measurement such as "length" or "elapsed time" become reified with frequent use.

This meaning of quantity comes with an imagined measurement process, and I attribute to that measurement process several components. Specifically, in order of a quantity to exist in the mind of a person, that person must also have a mental image of an *object*, meaning something to be measured -- such as a piece of paper; a *property* of that object -- such as the length of the longest side of the piece of paper; an act of *measurement* that produces the quantity, which includes choosing a *unit* to measure the property of the object with, and a *number* or *quantity* which is the result of that measurement.

For ease in making distinctions, I will refer to a number or a variable which is not the result of a measurement as *a value*. Also note that in common English, the term *measurement* can be used to refer both to an act of measurement and the numerical result of the measurement. I will use the term *measurement* only to refer to the former, and the term *quantity* to refer to the later.

The following framework however, deals only with a static mental image. If one imagines a dynamic situation, in which quantities are changing, an additional phrase is needed. For this purpose, I use the term *quantity* to refer to a changing numerical result, and the phrase *value of a quantity* to refer to the numerical result at a specific point in time, as one imagines the quantity changing.

This control mechanism is analogous to the checking phase of Carlson and Bloom's Multidimensional Problem Solving (MPS) Framework (Carlson & Bloom, 2005). The work of these authors presents a detailed look into the process of problem solving as performed by twelve mathematicians, which were all described as experienced problem solvers (either Ph.D. or near graduating with Ph.D.). This investigation resulted in the MPS Framework, which can be used for investigating, analyzing, and explaining mathematical behavior. In their telling of the problem solving cycle, the authors identify the phases of: orienting, planning, executing, and checking . Furthermore, it was determined that after orientation, the plan—execute—check cycle was often repeated throughout the problem-solving process. Also, within the planning process, the problem solvers were often determined to engage in a conjecture—imagine—verify cycle. This differs from the plan—execute—check cycle mainly because the former is much more informal than the latter. In other words, the former may be strictly limited to mental actions that do not necessarily result in formal products. One can think of this as the process in which a problem solver attempts to play out a solution in their mind in order to determine if it is appropriate or not to follow this "path" formally.

The orienting phase of the cycle is the phase in which sense making, organizing, and constructing takes place. This involves determining what the problem is asking and may include the defining of unknowns, sketching of a graph, constructing a table, etc. The planning phase involves making conjectures about the solutions approach. During this phase, the sub-cycle of conjecture—imagine—verify takes place. This sequence can be defined as follows: a) construct a conjecture, b) imagine how the solution will play out either through verbalization or silence (mental actions), c) evaluate the viability of the conjectured approach. This sub-cycle allows the problem solver a more efficient approach to the problem because it avoids a formal approach to each conjecture. The executing phase involves the problem solver making formal constructions and carrying out computations. The checking phase can be described as a process of verification. Here, the problem solver analyzes the reasonableness of their solution and computations. This results in a rejection (and a cycle back to planning) or an acceptance (an a move to a new problem solving cycle if not completed). In addition to just checking one's solution, problem solvers also often reflected on the effectiveness and efficiency of their decisions and actions.

We conjecture that this population of students is having difficulty in the orienting phase of problem solving. Therefore, we will be focusing aspects of our data collection on understanding the mental operations students engage in when processing the text in a problem context both in class and when completing problems in a clinical interview setting. In the clinical interviews, the situations presented required the identification and use of covarying quantities. Thus, we expected a limited ability of covariational reasoning to have a negative impact on a student's ability to properly complete the orientation phase. Here, the word "properly" is meant to describe the process of correctly orienting oneself in the problem such that one is able to make sense of the situation at hand and organize the information given. Specifically, this paper focuses on two aspects of the orienting phrase: forming a mental image of the problem situation, and quantifying aspects of that mental image.

Methods (Castillo-Garsow & Moore)

For the purposes of studying the covariational reasoning abilities of these college algebra students, we selected an interviewed 10 students from a college algebra class that was videotaped regularly. Five interviewers interviewed two student each, following a structured task-based clinical interview protocol. One of these interviews is the subject of analysis in this paper.

Following the classification put forth by Goldin (2000), "task-based" refers to a focus on specific mathematical problems which the student were asked to reason through and solve. During this process of attempting a solution, the students made use of their own mathematical reasoning, and the students were asked to verbalize their thinking, as well as his or her rationale for the mathematical choices that he or she makes. This talk-aloud task-based protocol is similar to the approach in follow Carlson and Bloom (2005).

Continuing to follow Goldin's (2000) classification, these task-based interviews were "structured" in that they will be designed to anticipate and control a number of factors in the interview. In addition to the mathematical task, possible student responses are anticipated and planned for, and the design of follow up questions is based on these anticipations. This approach was used so that results from interviews could be compared. However, this paper discusses only one student's interview.

In order to reveal as much student thinking as possible, our approach to conducting the interview, was to follow that of the method described by Goldin (2000). Each task-based interview was conducted such that the exploration into the students' understandings unfolds in four stages. First, the question was posed such that the student could respond freely. Thus, only nondirective follow-up questions were used initially(e.g., "Can you tell me more about that?"). This is denoted free problem solving. The second stage included minimal heuristic suggestions. For instance, if the response was not spontaneous, "Can you show me using some of these materials?" This may include directing a student to use or draw a graph or pictorial situation. Thirdly, the interview would make the use of guided heuristic suggestions. An example of this stage would be the question, "Do you see a pattern in the graph?" The last stage included the use of metacognitive, exploratory questions. This included questions such as "Do you think you could explain how you thought about the problem?" Although each stage is important in the exploration of student understandings, it is noted that the first stage must not be missed. By allowing a student free problem solving time, the student is able to act in a manner that is not guided or influenced by interviewer questions.

Tasks

A total of four problems were selected for this interview, along with accompanying questions, and they are included here in the order that they were presented to the students:

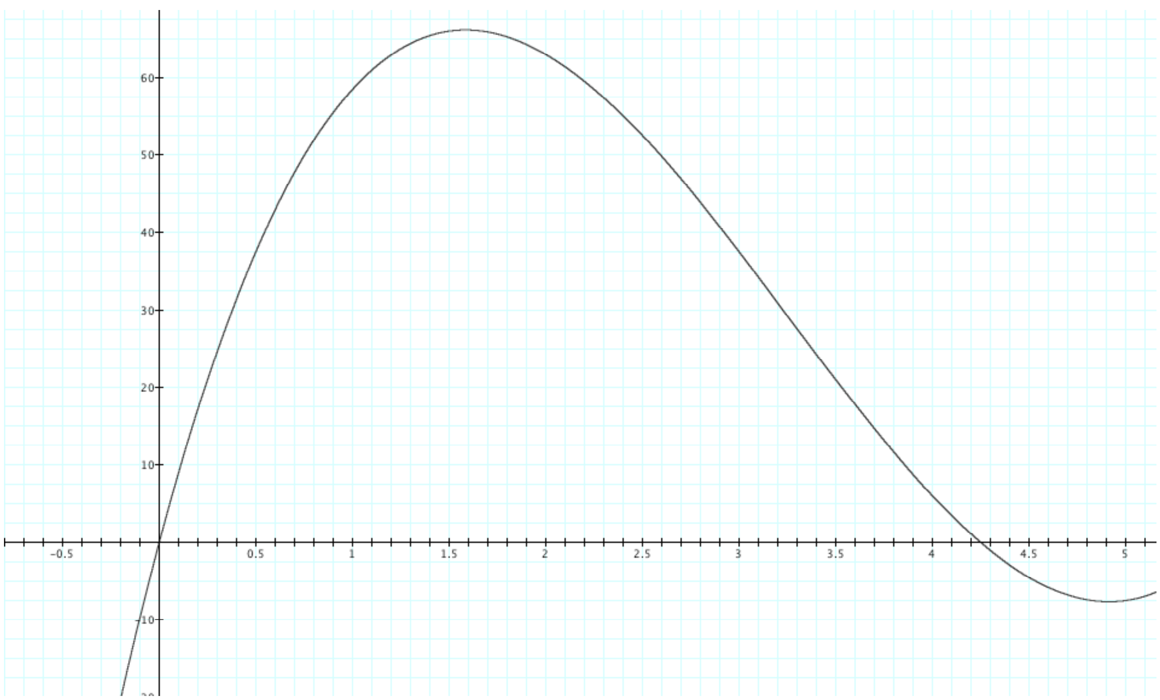
The Box Problem

Starting with an 8.5” x 11” sheet of paper, a box is formed by cutting equal-sized squares from each corner of the paper and folding the sides up.

Task 1: Describe to me how the length of the side of the cutout and the volume of the box covary.

Task 2: Write a formula that predicts the volume of the box from the length of the side of the cutout.

Task 3: Given a graph, describe how you would use this graph to describe how the volume changes as the length of the side of the cutout varies from 1.8 inches to 1.9 inches.



The box problem was one that all the students in the college algebra class were familiar with. The students interviewed had worked with this problem as part of a whole class discussion as well as as a homework assignment. The box problem addressed the orientation phase of problem solving, in that the student must picture how the paper is cut and folded in order to form a box, and how that picture may be quantified in order to establish a relationship between the length of the side of the cutout and the volume of the resulting box.

In addition, the box problem addressed the main ideas of covariation explicitly. By asking students to describe how the length of the side of the cutout and the volume covary in both task 1 and task 3. Task 1 is a free form question, in which students were expected to use whatever mental actions they were capable of. Task 3 directly addressed amounts of change, and the student's ability to engage in Mental Action 3.

The first two tasks had associated followup questions, which were to be asked as the student had difficulties:

Followup Questions for Task 1

1. “Say more.”
2. “Can you show me with the fingertool what you are thinking?”
3. “So for any length of cutout, the volume does ____, right?” (To be used, for example, if the student says something like “As the cutout gets bigger, the volume increases more and more”.)

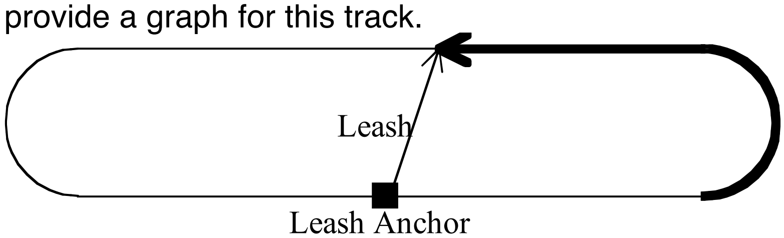
Followup Questions for Task 2

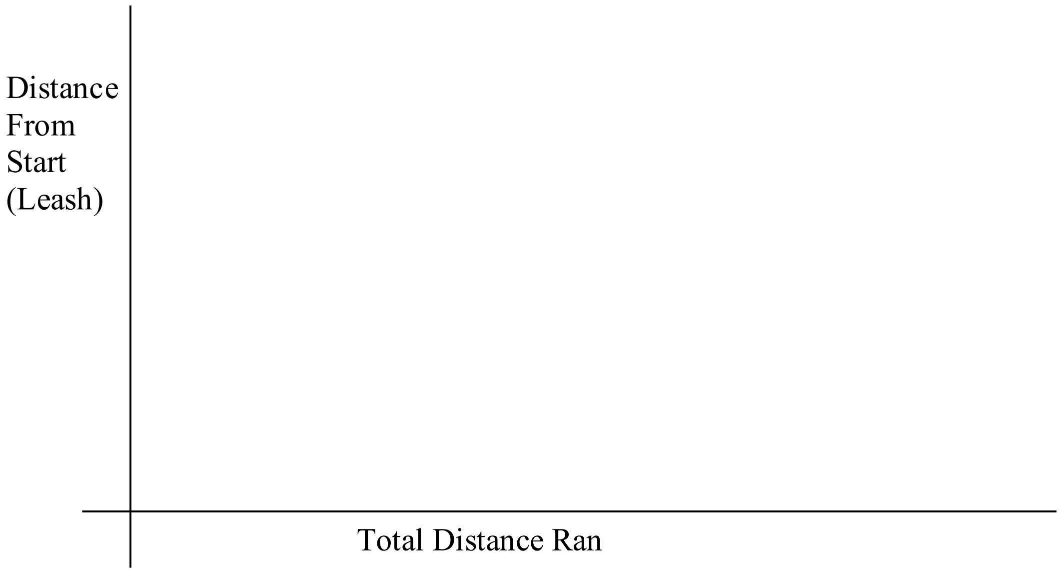
1. “Say more.”
2. “Describe the situation in your own words.”
3. “Represent this situation with a drawing.”
4. Give the student the drawing of a net from the following page, if the student was unable to construct a useful drawing in question 3.
5. “What are the quantities involved in this situation?”
6. “Where are these quantities on your drawing?”
7. “How do you find the volume of a rectangular box?”
8. “How would you calculate the _______ (length, width, height) in this situation?”
9. At the end, assuming the student has arrived at a formula for the volume, request: “Explain to me the parts of your formula.”

The Track Problem (Castillo-Garsow & Moore)

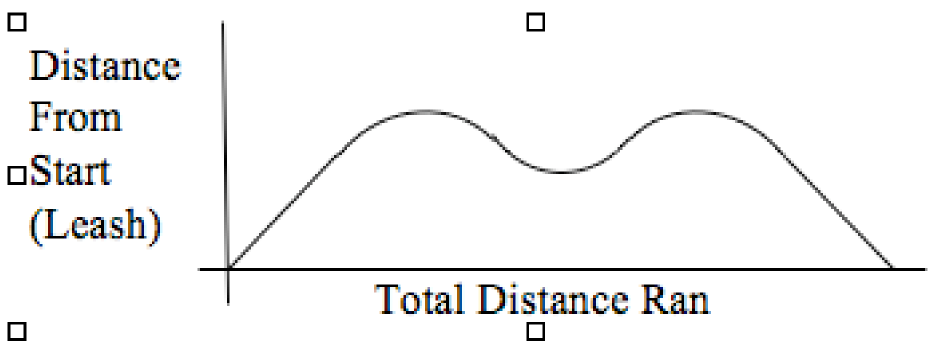
Pet Services Inc. has decided to expand their pet exercise ground to include multiple running tracks. In order to avoid pets escaping, the pets will remain on a leash as they navigate around the tracks. It has been determined that the pets are able to run most naturally if the leash remains taut. This will be accomplished by having each dog attached to a leash that is retractable. The anchor of the leash will be located at the start of the track.

In order to program the computers that control the length of the leash, we need a graph that shows the length of the leash given the distance the dog has ran around the track. Below, we have provided a drawing of one of the tracks to be used. Please provide a graph for this track.





Solution



This problem presented a covariational situation and asked for a graph of two covarying distances. This problem presented a situation in which a student had the option of using any covariational mental actions, and MA2-MA5 can be used to answer the problem correctly. This problem presented a simple situation (e.g., walking around a track), but at the same time required that the student process how the two quantities are covarying. Although this problem could be "solved" while only using language associated with MA5, we anticipated the need to determine if the subject how able to justify their solution by using behaviour associated with MA1-MA4 of the covariation framework. For instance, this may involve the drawing of line segments as the dog is running around the track and discussing how these line segments are represented in the graph (MA3-MA4).

In designing this task, we anticipated several difficulties that the student might have: First, they may struggle orienting themselves with the two covarying quantities and determining exactly what the quantities represent. For instance, they may not interpret what the distance from the starting point (e.g., the leash) represents. Rather than representing the leash, they may interpret that phrase to also mean the distance that the dog has run. Second, students may suggest that the distance from the start varies at a constant rate (first positive and then negative) with respect to the distance traveled. Students may also reverse the input and output on the problem. Another important student misconception is the thought that they must attend to the rate at which a dog runs around the track.

In order to determine at what level the student could reason, the interviewer was required to prompt the student to explain her/his thinking. This was done through subsequent questioning (again giving the task a “structured” form).

Followup Questions
What is the question asking of you?
Explain the situation in your own words.
What two quantities are varying and how are they represented in the situation?
What does each distance represent?
How does your graph describe the situation?
Can you describe this using the two quantities involved?
Ask the student to identify landmarks on the graph and interpret these in the context of the problem.
Ask the student to identify multiple points on the graph and relate this to the situation.
How long is the leash when the dog is halfway around the track?
Can you identify this both on the graph and on the situation?
Could you describe to the dog company how they may be able to determine the graph for any track they have?

The Train Problem (Bowling & Castillo-Garsow)

A mile-and-a-half-long train enters a tunnel at a constant speed of 20 miles per hour. The tunnel is five miles long. A clever thief is sitting in the front of the train, and he plans to rob a safe located in the back of the train and escape out the back without being seen. How fast must the thief walk (or run) to reach the safe under cover of darkness?

A Solution:
The thief wants to reach the rear of the train before the rear of the train exits the tunnel. Let R be the distance between the rear of the train and the end of the tunnel. The rear of the train must travel 6.5 miles before it exits the rear of the tunnel. Since the train is traveling at 20mph, R=6.5-20t. The rear of the train will reach the end of the tunnel in 6.5/20 hours, or 19.5 minutes.

The thief is walking 1.5 miles at at an unknown speed s. 1.5=st. The thief must walk that 1.5 miles in 6.5/20 hours in order to reach the rear of the train in time. This means the thief must walk at least 1.5/(6.5/20) or 4.62 miles per hour.

The train problem was used to focus on the orienting phase of problem solving in that there are many things that the students must make sense of in order to complete this problem: including interpreting the written problem, the relative motions of the train, the thief, and the tunnel; the choice of measuring from then beginning or the end of the train or the tunnel; The choice of frame of reference for those measurements (Measuring from the thief to the end of the tunnel is much more complicated than measuring from the thief to the end of the train).

The train problem also addresses covariational reasoning by introducing a large number of changing quantities to the situation, which the students must choose to attend to or to disregard. Possible pairs of covarying quantities include “Time since the front of the train entered the tunnel and the distance from the thief to the front of the tunnel” or “Time since the end of the train entered the tunnel and the distance from the end of the tunnel to the front of the train.” The selection of these two pairs of covarying quantities is unlikely, but they are used here to illustrate the range of possibilities that the student has to work with. The student’s actual selection of which quantities to relate will reveal how the student perceives a quantity and the formalization of a quantity, as well has how the student perceives a relationship between quantities and the formalization of those relationships.

In terms of Carlson’s mental action framework (Carlson et. al. 2003), the train problem requires Mental Actions 1 and 2: coordinating changes in values, and the direction of change; along with providing potential for other mental actions (coordinating, amount of change, average rate of change, and instantaneous rate of change) to surface.

Based on previous undocumented student attempts to work this problem, we anticipated certain difficulties. Specifically, students exhibited a difficulty in attending to the end points of the train and the tunnel as reference points. A common incorrect solution to the problem was to consider only the distance that the front of the train needed to travel (five miles), giving the thief only 15 minutes to complete his task.

A second, related difficulty was the students weak choices in documenting the quantities that they were working with. Students would often talk about quantities such as “The time” or “The train’s distance” without attending to where these quantities were being measured from. This was addressed explicitly in the interview by asking students to be clear about which time or which distance they mean.

<u>Followup Interview Questions:</u>
-Explain the situation in your own words.
-What is the problem asking you to figure out?
-Draw a picture of the situation.
-Draw a picture of the situation at a different time.
-What quantities are changing in this situation?
-Where are you measuring that distance from?
-What exactly is happening in the (15 minute) time period that you’ve found?

<u>The Fox and the Greyhound</u> (Castillo-Garsow & Postelnicu)

A greyhound is chasing a fox, which is 40 fox leaps ahead of him.

While the fox makes 9 leaps, the greyhound makes only 7 leaps, and 3 greyhound leaps are like 5 fox leaps.

a) Is it possible for the greyhound to overtake the fox?
b) If not possible, justify why not.
c) If possible, after how many leaps does the greyhound overtake the fox?

The greyhound and the fox problem involves the covariation of five quantities: The number of leaps taken by the fox, the number of leaps taken by the greyhound, the distance traveled by the fox, the distance traveled by the greyhound, and the time elapsed. However, a fox leap or a greyhound leap can each be used as a unit of time or a unit of distance. This reduces the number of quantities in the problem to three: The distance traveled by the fox (measured in fox leaps or greyhound leaps), the distance traveled by the greyhound (measured in fox leaps or greyhound leaps), and the elapsed time (measured in fox leaps or greyhound leaps). This problem directly addresses Carlson's mental action 3, in that the student is being asked to coordinate amounts of changes in time, fox distance, and greyhound distance, measured in units of fox leaps and greyhound leaps.

This problem relies upon recognizing that the proportional relationship between fox leaps and greyhound leaps describe relationships in distance and time, and that although there are two explicit quantities in the problem (number of fox leaps and number of greyhound leaps), there are three quantities in the situation: The distance traveled by the fox (measured in fox leaps or greyhound leaps), the distance traveled by the greyhound (measured in fox leaps or greyhound leaps), and the elapsed time (measured in fox leaps or greyhound leaps). Followup questions were designed to help the student orient to the quantities involved in the problem:

Followup Questions
a) What is this problem about?
b) Can you describe the chase?
c) Would you say that the fox is faster than the greyhound?
d) Can you compare the fox leap with the greyhound leap?
e) What is your meaning for “while the fox makes 9 leaps, the greyhound makes only 7 leaps”?
f) What is your meaning for “3 greyhound leaps are like 5 fox leaps”?
g) Is it possible for the greyhound to overtake the fox?
h) Can you imagine that the fox and the greyhound start from the same place, at the same time, running in the same direction? Describe the race.
i) Can you say who will win the race?
j) We know that “while the fox makes 9 leaps, the greyhound makes only 7 leaps”.

While the fox makes 18 leaps, the greyhound makes…how many leaps?
While the fox makes 27 leaps, the greyhound makes …how many leaps?
What do you notice?

k) We know that “3 greyhound leaps are like 5 fox leaps”.

Six greyhound leaps are like… how many fox leaps ?
Nine greyhound leaps are like… how many fox leaps?
What do you notice?

l) Can you imagine now the fox and the greyhound starting from the same place, at the same time, running in the same direction? Describe the race.

m) Would it be helpful to have only greyhound leaps or only fox leaps to describe the race? How can you do it?

n) During the time necessary for the greyhound to make 21 leaps, the fox makes …how many leaps? Twenty-one greyhound leaps are like…how many fox leaps? What happens with the distance between the greyhound and the fox?

o) What do you think now? Is it possible for the greyhound to overtake the fox?

p) If the fox is 40 leaps ahead, after how many "instances/frames" like the one we described before, will happen that the greyhound overtake the fox?

q) After how many leaps does the greyhound overtake the fox?

Analysis (Castillo-Garsow & Moore)

Clinical interviews offer the testing and development of theoretical hypotheses (Clement, 2000). Theoretical hypotheses attempt to explain why correlations seen in the subjects' actions occur. For instance, the covariation framework described above can be considered a hypothesis used to describe various subjects' mental actions relative to covariational reasoning. The nature of the testing or development of theoretical hypotheses can either be convergent or generative. A convergent purpose is to provide empirical findings and test hypotheses. This often involves the coding of interactions through a pre-determined "lens" (e.g., Carlson's covariation framework). Generative purposes are conducted in order to generate new observation categories or theoretical models. This calls for an interpretive analysis and is often the source of grounded theory (Strauss & Corbin 1998).

The interview that is the subject of this paper was transcribed, and the transcription was subject to microanalysis and open coding, as described by Strauss and Corbin. As a method of microanalysis, I analyzed each statement made by Kim on the word and phrase level, with an eye for how she must have been reasoning in order for her statements and behaviour to be rational actions. This method of analysis is described by von Glasersfeld (1995; Thompson 2000) as *conceptual analysis*, which he describes as "what mental operations must be carried out to see the presented situation in the particular way one is seeing it" (von Glasersfeld 1995, p.78).

This microanalysis was then used to assign codes to the transcript, where the codes were summaries, observations, notes, and conclusions pointing to similarities and recurrent themes in the text. This open coding (Strauss & Corbin 1998), was the basis for the final analysis of the text, when I observed that the open codes fell into three primary categories: quantification, mental image, and covariation, each with a number of subcategories described in detail in the results section. This method of hierarchically categorizing the codes was not originally intended to be part of the analysis, but emerged from necessity and patterning in the data. The choice of method, however, was influenced by Strauss & Corbin's axial coding methods.

The Interview

Subject

For this study, I interviewed "Kim." Kim was one of about 10 students who volunteered to participate in the study. My choice of Kim as the subject for my study was based on her availability coinciding with times when I was also available to conduct the interview. She was compensated for her time. According to her course instructor, "Kim" is a strong student in the class. She currently has a grade of B+ and has completed all of the homework assigned in the course. The interview lasted approximately an hour and a half, but the recording stopped after an hour and 13 minutes. The hour and 13 minutes that was recorded was transcribed and analyzed.

Overview

Kim began with task one of the box problem (lines 59 – 230), where she initially concluded that changing “the length” would not affect “the volume” because the amount of paper would stay the same. We discussed how the paper folded into a box, and what was causing the length to change. After drawing her attention back to the problem statement and the fact that pieces of the paper were being cut out, Kim concluded that as “the length” decreased, “the volume” would decrease, because there would be less paper.

For task 2 of the box problem (lines 235-436), Kim had difficulty remembering the formula. After making a list of quantities, and identifying them on her diagram, Kim was able to articulate how the paper folded up into a box. She was also unable to remember the formula for the volume of the box. After being giving her the formula, volume equals length times width times height, Kim constructed a formula for the volume of the box based on the length of the cutout: Volume = (11 – cutout) (8.5 – cutout) cutout. She briefly debated between (11 – cutout) and (11 – 2 * cutout), but settled on (11 – cutout) because removing one cutout from each side would go all the way around the perimeter of the paper without double-counting cutouts (four sides and four cutouts, so subtract one cutout from each side).

For task 3 of the box problem (lines 476-569), Kim found the points 1.8 and 1.9 on the x axis, followed them up to the curve, and concluded that as “the length” increase, “the volume” decreases. Kim then expressed concern that this conflicted with her conclusion in task 1 that as “the length” decreases “the volume” decreases She resolved this conflict when she realized that the graph in task 3 was asking about “the cutout” while her conclusion in task 1 was about “the length” (presumably the length of the side of the box). I then followed up by asking her about an interval of length of cutout in which the volume increases, and she revised her answer in task 1 to conclude that as “the cutout” increases, “the length” decreases, and that as “the length” decreased, “the volume” would first increase, and later decrease.

In the track problem (lines 597-887) Kim initially had difficulty with the problem because she imagined the leash as following the path of the dog, rather than cutting across the track. Once we agreed on the behavior of the leash, Kim solved the problem by finding local maxima (furthest distance and highest point on the graph) and minima and connecting them to form a reasonably accurate graph.

In the train problem (lines 901-1572) Kim began by rephrasing the problem in her own words and drawing a sketch of the situation, without prompting. Her first goal was to calculate how long the train would spend in the tunnel, but she had difficulty with the fractions involved. Kim did not see that the train could cover 6.5 miles and still keep the thief in darkness. Kim and I had a long discussion on how she imagined measuring distances, the results of which are described in a later section. Afterward, Kim concluded that the train would travel five miles in a quarter of an hour. She then tried to figure out the speed the thief would need to go by using a formula which she couldn't recall. Rather than helping her recall the formula, I focused on her understanding of the situation and her understanding of distance.

In the fox and greyhound problem (lines 1592-2108), Kim took the lead, with myself asking only very few questions. She initially had difficulty interpreting the problem, until she realized that one of the relationships described in the problem involved distance, and one didn't (she never specified that the other relationship involved time). She found the number of fox leaps that would be equivalent in distance to a greyhound's seven leaps, although she had difficulty with the fraction, until I suggested that she just leave it as a fraction. By comparing the two distances measured in fox leaps, she concluded that the greyhound would eventually catch up. By counting multiples of these distances, she found the number of greyhound leaps it would take for the greyhound to catch up (some of the final calculations are missing in incomplete recording).

Quantification

One of the theoretical perspectives that I kept in mind during the analysis was that in order to orient to a problem space mathematically, it is necessary to quantify aspects of the situation. Kim's difficulties with quantification illustrated several of the important aspects of quantification as a part of orienting to a problem space.

Distinctions in Quantification

During the interview, Kim did not make distinctions between an object, a property of that object, and the quantity associated with the measurement of that property. The following examples were extracted from Kim's work on the Box problem:

***** KIM *****
[369][697.189 Kim] Well, your 8.5 is your width
[370][700.486 Kim]
[371][702.182 Kim] maybe the a- I'm confused now
[372][705.123 Kim] ok...
[373][709.204 Kim]
[374][713.652 Kim]
[375][716.455 Kim] does it, it doesn't
[376][718.27 Kim] the volume will depend on
[377][720.856 Kim] what the size of the cutout is, so
[378][723.239 Kim]

***** -CARLOS-KIM *****
[379][724.91 1] ok
[---][724.91 2] it would be like

***** KIM *****
[380][726.246 Kim]
[381][728.778 Kim] 11 minus your cutout
[382][730.864 Kim] oh, I remember this now, kind of

On line 369, Kim assigned a number 8.5, to a property of an object (width), but did not assign units to the number. This suggests that she was engaging in an act of measurement. Also, the choice of the verb "is" is significant here, in that it assigns a value to a property, not to the measurement of that property. Here, Kim is using "width" as a variable, and she says "8.5 is your width" in much the same way that someone might say "x is 5." This is significant because 8.5 is not the value of the width, but rather a relationship between with width and the chosen unit.

Also note here that Kim does not mention the object that she is measuring. The width is not a width of something, but rather an object in and of itself. This becomes problematic for her during this problem and others, whenever there are multiple widths and lengths.

"width" then, for Kim, does all the work of the object, the property of the object, and the numerical result. To Kim, "width" is an object, and it has a value of 8.5.

On line 381, Kim uses the "cutout," an object, in a numerical operation (subtraction from 11), indicating a similar type of blurring of object, property, and quantity, although Kim does make a rudimentary distinction between object and property (on line 377, she refers to "the size of the cutout"). However, for her, size does not appear to be a property that can be measured. On line 381, she chose to say "11 minus your cutout" not "11 minus the size of the cutout."

Another example can be seen in this excerpt:

***** KIM *****
[198][371.535 Kim] Ummm, well if you vary

[199][373.334 Kim] If you cut out, like the corners
[200][375.355 Kim] like an inch or so, it'll make the box smaller
[201][378.727 Kim] depending on like what you cut out

Here, on line 200, Kim talks about the box becoming smaller. But the box becoming smaller could mean any number of things. Is Kim referring to the surface area of the box? Is Kim referring to the volume of the box? If Kim is referring to the surface area of the box, then cutting out larger corners would indeed make the surface area of the box smaller. But the problem is asking about the volume of the box, and the volume of the box doesn't always get smaller as corners are cut out.

Here, it appears that Kim is conflating two properties of an object (the volume and surface area), and treating them as a single quantity. A few earlier examples also support that she may be conflating surface area and volume:

[72][139.168 Kim] It won't matter will it, if the length
[73][141.521 Kim] of the side varies, then the volume's gonna be the same, cause
[74][144.74 Kim] it's the same amount of paper

Kim continues:

[88][171.895 Kim] The volume's gonna be the same
[89][173.914 Kim] because
[90][175.053 Kim] I think, because like
[91][176.937 Kim] The sheet of paper, if you're using the whole sheet
[92][179.886 Kim] doesn't the length won't
[93][182.241 Kim] matter cause the volume's like
[94][183.82 Kim] inside?

Also, shortly after the "box smaller" comment.

[213][402.189 Kim] if you cut large
[214][404.407 Kim] like
[215][405.599 Kim] corners off the paper and folded it then the volume would get
[216][409.01 Kim] smaller
[217][409.782 Kim] cause the box would be smaller

Here, Kim makes a distinction between the volume (line 215) and the size of the box (line 217), and she also makes a distinction between the volume (lines 73, 93-94) and the amount of paper (lines 74, 91), but she does not distinguish them as quantities. As the size of the box does, or as the amount of paper does, so does the volume. If the amount of paper (surface area) increases, the volume will also increase. If the amount of paper decreases, the volume decreases. If the amount of paper stays the same, then no matter how else the paper changes, the volume stays constant, This indicates a conflation of the ideas of surface area and volume.

Lack of measurement

In line 377 above, Kim refers to "the size of the cutout" as if it were a quantity. However, size is not a property of an object that can be measured. The size of the cutout could refer to the square cutout's length, area, weight, mass, or thickness; just to name a few properties. This ambiguity of phrasing could be taken to mean one of two things: Either that she is not imagining the act of measurement, or that she considers the image of measuring in her mind to be so obvious that "size" could not possibly refer to anything else.

This brings up the question of how much Kim is imagining the act of measurement. Take this earlier example:

[72][139.168 Kim] It won't matter will it, if the length
[73][141.521 Kim] of the side varies, then the volume's gonna be the same, cause
[74][144.74 Kim] it's the same amount of paper

Here on lines 72-73, Kim refers to the length of the side changing, but on line 74, she refers to the "amount of paper" not changing. Unfortunately, it's not clear what Kim means by the amount of paper, or which side she is referring to (the length of the side of the paper, the length of the side of the box, or the length of the side of the cutout). However, in the context of the problem statement -- corners cut out of an 8.5 by 11 inch piece of paper -- it is clear that Kim is not imaging how changing the lengths would affect the amount of paper. More specifically, she is not imagining the act of measuring the length nor the act of measuring the amount of paper, as either of those would lead her to understand that changing the length of any of these sides *does* change the amount of paper.

Properties without objects

Earlier, I discussed Kim's speech as indicating that "width" was an object in and of itself, as was length -- rather than being a width of an object or a length of an object. Her image of width and length as objects rather than as measurements appears to have the following consequences.

[356][674.956 Carlos] So the volume of a box is length times width times height

***** KIM *****
[357][678.473 Kim] That was it, yes ok.
[358][680.444 Kim] I got it now

[...omitted for brevity...]

***** KIM *****
[369][697.189 Kim] Well, your 8.5 is your width
[370][700.486 Kim]
[371][702.182 Kim] maybe the a- I'm confused now
[372][705.123 Kim] ok...

Here, Kim is trying to find a formula for the volume of the box (BOX task 2), however, rather than using the width of the box for the width in the formula, she initially focuses on the width of the paper (8.5 inches).

Throughout the box problem, Kim consistently confused the length of the side of a cutout, the length of the side of the box, and the length of a side of the paper. This confusion is illustrated in the following:

[64][117.586 Kim] Task one describe to me how the length of the side of the cutout and the volume of the box covary
[65][122.676 Kim] so
[66][124.464 Kim] How do I even do this?
[67][127.806 Kim]
[68][131.042 Kim] Well as the length
[69][132.37 Kim] varies
[70][133.596 Kim] The volume's gonna vary er
[71][136.001 Kim]
[72][139.168 Kim] It won't matter will it, if the length
[73][141.521 Kim] of the side varies, then the volume's gonna be the same, cause
[74][144.74 Kim] it's the same amount of paper

[...omitted for brevity...]

[88][171.895 Kim] The volume's gonna be the same
[89][173.914 Kim] because
[90][175.053 Kim] I think, because like
[91][176.937 Kim] The sheet of paper, if you're using the whole sheet
[92][179.886 Kim] doesn't the length won't
[93][182.241 Kim] matter cause the volume's like
[94][183.82 Kim] inside?

On line 64, Kim reads "the length of the side of the cutout," But by line 72-73, this has already become "the length of the side." Whether the length of the side here refers to the length of the side of the cutout, the length of the side of the box, or the length of the side of the paper is unclear. By line 92, "the length of the side" has become simply "the length," And Kim uses "the length" to refer to both the length of the box and the length of the paper on various occasions. When asked to label "the length" on her diagram, she chose the length of the side of the box:

[280][541.841 Carlos] length and cutout from corners, and volume
[281][544.423 Carlos] umm could you like could you label those on your picture?

On line 425, she refers to "the length" as being the length of the side of the paper
[425][816.008 Kim] the 11 is the length
[426][818.835 Kim] so to find the like
[427][821.165 Kim] length of the box

When the length of the cutout is reintroduced in Task 3, Kim confuses it with her previous results using "the length."

[476][921.273 Kim] given the graph describe how you would use this graph to describe how the volume changes as the length of the size of the cutout varies from 1.8 inches to 1.9 inches.
[477][931.002 Kim] so
[482][942.06 Kim] 1.8
[483][943.704 Kim]
[484][946.195 Kim] is the length so I think that's
[485][948.274 Kim] on the x so
[486][950.044 Kim] find it on here
[487][951.356 Kim]
[488][953.546 Kim] which is right here
[489][954.703 Kim] it's 1.8 and then 1.9's right here
[490][957.46 Kim] so as the cutout varies
[491][959.89 Kim] from 1.8 inches to 1.9 inches it
[492][962.117 Kim] the length
[493][963.625 Kim]
[494][965.09 Kim] the volume decreases

Here, on line 484, Kim refers to the length of the cutout as "the length" This leads her to question her earlier conclusion that as "the length" decreases, the volume decreases.

TASK 1:
***** KIM *****
[211][397.989 Kim] It would change
[212][400.225 Kim] it would- if you cut smaller
[213][402.189 Kim] if you cut large
[214][404.407 Kim] like
[215][405.599 Kim] corners off the paper and folded it then the volume would get
[216][409.01 Kim] smaller
[217][409.782 Kim] cause the box would be smaller

AFTER TASK 3:

***** KIM *****
[502][977.846 Kim] so that's [task 1] wrong.
[503][980.262 Kim] right?

***** -CARLOS-KIM *****
[504][981.127 1] Why is that wrong
[---][981.127 2] as the length

***** KIM *****
[505][982.411 Kim] increases, the volume
[506][984.496 Kim]
[507][985.648 Kim] cause the length is increasing here
[508][987.971 Kim] so then the volume decreases
[509][990.193 Kim] so this would be the opposite

***** CARLOS *****
[510][991.539 Carlos] so as the length increases, the volume decreases?

Kim eventually resolves this difficulty by calling one length (the length of the side of the box) "the length;" and the other length (the length of the side of the cutout) "the cutout"

***** KIM *****
[524][1013.157 Kim] oh wait, this is the cutout
[525][1014.678 Kim] wait, now I'm confused
[526][1016.76 Kim]
[527][1018.83 Kim] cause the cutout increases, the length will decrease
[528][1021.531 Kim] so no, that's [task 1] right then
[529][1023.073 Kim]

<u>Distance as a location</u>

Related to issues of quantity and measurement is Kim's concept of distance. When asked to imagine the act of measuring distance, Kim imagined measuring distance not as a length, but as a location in a coordinate system. The excerpts that follow are from the train problem. In both excerpts, Kim is describing passing through the tunnel, at the end of the trip. :

***** KIM *****
[1367][2775.717 Kim] and then you would stop
[1368][2777.259 Kim] once it hits there

[1369][2778.595 Kim]
[1370][2780.653 Kim] stop measuring, I guess

[...omitted...]

***** CARLOS *****
[1526][3098.129 Carlos] And I'm looking at
[1527][3100.339 Carlos] again, just the front of the train

***** KIM *****
[1528][3102.499 Kim] It would be zero because it's not in the tunnel
[1529][3104.967 Kim] but it's gone
[1530][3105.892 Kim] yeah

***** CARLOS *****
[1531][3107.384 Carlos] so it would be zero?

***** KIM *****
[1532][3109.543 Kim] no, it'd be five

***** CARLOS *****
[1533][3111.497 Carlos] it'd be five.

***** KIM *****
[1534][3114.068 Kim] well
[1535][3114.685 Kim] it's gone five miles
[1536][3116.864 Kim]

On line 1370, Kim said that when the train exits the tunnel, you "stop measuring", on line 1528, Kim explained what "stop measuring" means, That the distance-location is zero because the train is no longer in the tunnel. Another example of this can be seen as Kim described measuring the distance traveled by the thief on the train:

[1136][2305.247 Kim] the thief's just moving on the train
[1137][2307.817 Kim]
[1138][2309.924 Kim] so as
[1139][2310.797 Kim] as he moves the distance will
[1140][2312.751 Kim] change
[1141][2313.676 Kim] cause he's moving along the train which is a mile and a half.

***** CARLOS *****
[1142][2316.759 Carlos] and so if I did the same thing with the tape measure
[1143][2319.124 Carlos] where would I be holding the tape measure

[1144][2321.134 Carlos] to measure that distance
[1145][2322.265 Carlos]

***** KIM *****
[1146][2323.603 Kim] well he starts out in the front I think cause he said
[1147][2326.454 Kim] so he'd be here
[1148][2328.025 Kim] and then
[1149][2329.516 Kim] the end of the tape measure would be like here
[1150][2331.778 Kim] [Indicates the end of the train]

Kim described the distance that the thief has traveled as changing (lines 1139-1141), but when she described how to measure that distance with a tape measure, she placed the ends of the tape measure at the ends of the train (lines 1146-1150). For Kim then, a distance is composed of three points. Two endpoints of the trip to establish a coordinate system, and the location of the object as a point on that coordinate system.

Quantification of Time

Kim very rarely made use of explicit time when constructing her solutions to the problems. Instead she used other more tangible quantities as ways of keeping track of time. In the Fox and Greyhound problem, she never made an explicit reference to time. Instead, Kim kept track of time by using greyhound leaps, while simultaneously using fox leaps to keep track of distance. Unlike the example of the various lengths in the box problem, Kim never got her leaps confused. This except shows her solution to the problem:

***** KIM *****
[1920][3966.486 Kim] yeah
[1921][3968.749 Kim] that's why I'm being confused, because like fraction
[1922][3972.091 Kim] but
[1923][3973.684 Kim] so if the
[1924][3976.1 Kim] greyhound's going seven
[1925][3977.623 Kim] leaps
[1926][3979.679 Kim]
[1927][3981.427 Kim] it equals
[1928][3982.762 Kim] 10 plus five thirds
[1929][3984.612 Kim] fox leaps
[1930][3985.692 Kim]
[1931][3991.808 Kim]

***** CARLOS *****
[1932][3993.299 Carlos] ok
[1933][3994.43 Carlos]
[1934][3996.178 Carlos] and you got that from
[1935][3998.697 Carlos] this?
[1936][3999.613 Carlos] and that?

***** KIM *****
[1937][4000.333 Kim] mhm

***** CARLOS *****
[1938][4000.95 Carlos] ok
[1939][4002.029 Carlos] so 10 plus five thirds fox leaps

***** KIM *****
[1940][4004.444 Kim] which is going to be more than
[1941][4006.397 Kim] what the fox took

***** CARLOS *****
[1942][4007.63 Carlos] ok
[1943][4008.915 Carlos] ok
[1944][4010.162 Carlos] so because that's more than the fox takes

***** KIM *****
[1945][4012.328 Kim] eventually, the greyhound will catch up

Here, on line 1924, Kim used the seven greyhound leaps as an event, or as a unit of time. She calculated the number of fox leaps (distance), that the greyhound traveled in that time (lines 1927-1929). She then compared this to the number of leaps that the fox took in the same amount of time (lines 1940-1941), and concluded that the greyhound traveled a greater distance than the fox in the same unit of time (line 1945). Throughout this exchange, however, time is never mentioned.

In the train problem, Kim made explicit references to time in the form of hours and minutes. However, she also used distance as a variable to represent time, as illustrated in this example below:

***** KIM *****
[1274][2584.762 Kim] well, in the tunnel, the thief wants to move
[1275][2587.963 Kim] to the back of the train in that five miles.

In this brief excerpt, Kim uses the distance the train travels in the tunnel as a timer for the thief. She says that the thief has five miles worth of time to travel from the front of the train to the back of the train (lines 1274-1275).

This usage of distance as a time is reminiscent of the speed-lengths described in Thompson and Thompson (1994), In which speed is a distance, and distance is the number of times that that speed is laid end to end. In this structure, distance can easily be used as a time, because the number of times that the speed-length is used is the same number as the number of units of time.

Indeed a close analysis reveals this instance in which Kim explicitly refers to speed as a length.

[928][1840.569 Kim] I'm confused
[929][1842.369 Kim] We wanna find
[930][1843.759 Kim] The speed, which is in miles

It is not clear, however, that the speed-length is Kim's dominant paradigm for thinking about the relationship between speed, distance, and time, and the evidence that I have is not enough to support such a claim. The quote above is the only case in which Kim refers to speed as a length, or makes use of the image of laying speed-lengths end to end in order to count a time, as described in Thompson and Thompson. In the remainder of Kim's speech it's not clear if Kim is thinking in speed lengths or making use of proportional reasoning:

[1229][2489.455 Kim] 20 miles an hour
[1230][2490.999 Kim] and the tunnel's five so
[1231][2494.346 Kim]
[1232][2495.477 Kim] you have to divide that, I think but
[1233][2498.137 Kim] I don't know
[1234][2499.371 Kim]
[1235][2500.297 Kim] so there's only
[1236][2501.838 Kim]
[1237][2505.539 Kim] so the train is only five miles, which is a quarter of what it goes
[1238][2509.242 Kim] in an hour
[1239][2510.27 Kim]
[1240][2512.223 Kim]
[1241][2517.215 Kim] I don't know where to go from here

***** CARLOS *****
[1242][2519.206 Carlos] can you say
[1243][2521.52 Carlos] like quarter of what he goes, can you

***** KIM *****
[1244][2524.075 Kim] well
[1245][2525.618 Kim] ok
[1246][2528.652 Kim] I'll draw it, this is a square
[1247][2531.068 Kim] and say that's like a track
[1248][2533.792 Kim] of the train
[1249][2535.384 Kim] every
[1250][2536.874 Kim] well it's one hour, this whole thing represent hour so
[1251][2540.266 Kim] it would go five
[1252][2542.579 Kim]
[1253][2544.07 Kim] every like, fourth of an hour

Here, the result of "a quarter" could have come from thinking in speed-lengths (5 miles is a quarter of 20 miles), Or by reasoning about the proportional relationship between distance and time (a quarter of the distance means a quarter of the time). On line 1237, it appears that she is thinking in terms of speed-lengths, by relating the two distances, but on lines 1250-1253, Kim applies quartering process to the time. In speed-length thinking, time is a result of an operation (counting the number of speed-lengths in a distance). In relating the fraction of the distance to the fraction of the time (lines 1250-1253), Kim is extending beyond the limitations of thinking of speed as a length.

Forming Mental Images

Part of orienting to a problem space may involve forming a mental image of the situation: imagining what the situation might look like at various point in time, or how a person might interact with the situation if that situation were physical. When looking at the interview from this perspective, Kim had several difficulties with the problems that can be explained by the mental images of the problem situation that she formed or didn't form.

Problem before mental image

In the box problem, Kim began working before spending time forming a mental image of the situation.

***** KIM *****
[59][107.902 Kim] Oh, read it, sorry.
[60][109.051 Kim] Starting with an 8.5 inch by 11 inch
[61][111.691 Kim] sheet of paper
[62][112.514 Kim] A box is formed by cutting equal sized
[63][114.879 Kim] squares from each corner of the paper and folding the sides up
[64][117.586 Kim] Task one describe me how the length of the side of the cutout and the volume of the box covary
[65][122.676 Kim] so
[66][124.464 Kim] How do I even do this?
[67][127.806 Kim]
[68][131.042 Kim] Well as the length
[69][132.37 Kim] varies
[70][133.596 Kim] The volume's gonna vary er
[71][136.001 Kim]
[72][139.168 Kim] It won't matter will it, if the length
[73][141.521 Kim] of the side varies, then the volume's gonna be the same, cause
[74][144.74 Kim] it's the same amount of paper

In this excerpt, Kim read the problem (lines 59-64), and then her first concern was not sense making, i.e. "what's going on in this situation?," but rather "How do I do this?" (line 66). She begins reaching conclusions without forming a mental image of the situation (lines 72-74).

Evidence that Kim has not yet formed a mental image of the situation can be found in later excerpts, as her mental image of the situation begins to develop:

[125][238.546 Kim] There's a certain way you fold the paper, like
[126][240.952 Kim] I know you can make a box out of the paper, I don't know how to describe it, like you fold
[127][245.194 Kim]
[128][245.99 Kim] You fold up the sides and then
[129][247.959 Kim] fold it

In the above except, Kim is describing the way in which the paper folds to form a box, but she is uncertain about the mechanics of the folding. Part of that uncertainty comes from not attending to the existence of the cutouts. She is unaware at this time that the paper has corners cut out of it. In the next excerpt, Kim describes having done the same problem in class:

[159][303.749 Kim] We covaried the length
[160][305.114 Kim] Of the sides and made boxes and then that
[161][309.005 Kim] changed the volume

***** CARLOS *****
[162][310.119 Carlos] And why was the length of the side changing?
[163][311.804 Carlos]

***** KIM *****
[164][314.292 Kim] cause of the problem
[165][315.7 Kim] What do you mean?

***** CARLOS *****
[166][317.087 Carlos] wha-wha-what in the problem was making the length of the side change?
[167][320.077 Carlos]

***** KIM *****
[168][322.958 Kim] I don't understand

***** CARLOS *****
[169][324.969 Carlos] Ok

***** KIM *****
[170][325.482 Kim] Sorry

***** CARLOS *****
[171][326.627 Carlos] Umm
[172][327.414 Carlos] you said, you know

***** -KIM-CARLOS *****
[173][328.657 1] the length was changing
[---][328.657 2] here

***** CARLOS *****
[174][329.413 Carlos] the length isn't changing
[175][330.661 Carlos]
[176][331.435 Carlos] You know, in this diagram, the length isn't changing?
[177][334.149 Carlos] but in the problem you did in class, the length was changing

***** KIM *****
[178][336.791 Kim] Oh, cause we were told to change the length

In line 159, Kim describes "the length" changing, but she is not clear on what caused the length to change. On lines 164 and 178, she attributed the cause of "the length" changing to the problem. In none of these lines does Kim describe the act of cutting out the corners, or that that might affect the length. In this excerpt, "the length" changes just because that's what the problem says to do. She did not coach her explanation in terms of a measurement process: the object being measured, how that object was being changed, or that changes effect on the quantity of “the length.”

After I reminded her of of the cutout, I asked Kim to list the quantities involved in the situation, and locate those quantities on her diagram, Kim formed a different mental image of the situation:

***** KIM *****
[299][576.487 Kim] you
[300][577.841 Kim] cut out a certain
[301][579.536 Kim] like whatever - equal sizes around the
[302][583.119 Kim] paper whatever size - supposed to be and then
[303][586.519 Kim] You fold it up, fold the sides up together to make the box.

In contrast to the previous excerpts, in lines 300-301 Kim refered to the cutout, and described that cutout and it's effect on the length. In line 303, Kim explained how the sides fold to form a box, in contrast to lines 125-126, where Kim was unsure.

Kim began working on the problem on line 66, but she develops a mental image of the situation over the course of working on the problem. It is seven and a half minutes later, and after I ask her about the cutout, and quantities directly, she finally revealed the mental image that is exhibited in lines 299-303.

Impossible Mental Images

In both the Box problem and the Pet problem, Kim initially described situations that were physically impossible, indicating that she was working with a partially formed mental image. Put differently, the mental image that she had formed was not one of a physical situation that played out or was interacted with.

In the box problem, Kim's initial explanation of how the box was formed was physically impossible. It is reproduced here:

[125][238.546 Kim] There's a certain way you fold the paper, like
[126][240.952 Kim] I know you can make a box out of the paper, I don't know how to describe it, like you fold
[127][245.194 Kim]
[128][245.99 Kim] You fold up the sides and then
[129][247.959 Kim] fold it

As you may recall from the discussion in the previous section, this excerpt occurred before Kim's attention was drawn to the cutout. So, how might we imagine folding up the sides of a whole piece of paper to form a box? Forming a box out of a whole sheet of paper would either require a series of complicated origami fold -- not what Kim described, or matter would have to pass through itself in order to fold the sides up to form a box -- a physically impossible situation.

In the Pet problem, Kim initially began with an image of the leash following the dog around the track, rather than crossing the center of the space. This led to a physically impossible situation as well.

[609][1166.138 Kim] the leash
[610][1167.09 Kim] is going all the way around so the graph's gonna go up
[611][1170.766 Kim] increasing and then there's gonna hit a point and then
[612][1173.379 Kim] it's gonna start to decrease again
[613][1174.871 Kim]

***** CARLOS *****
[614][1175.59 Carlos] it's gonna start to
[615][1177.084 Carlos] why will it start to decrease

***** KIM *****
[616][1178.318 Kim] because once it hits the middle
[617][1180.412 Kim] it'll start to go back to start
[618][1182.475 Kim]

On lines 609-610, Kim describes the leash following the dog around the track. This was accompanied by hand gestures which are not included in the transcript. If the leash were to follow the dog around the track, then the length of the leash would be exactly the same as the distance the dog had traveled. But on lines 611-612, Kim described the length of the leash decreasing after a certain point. (The midpoint of the dog's travel, line 616). This is problematic, because in order for the length of the leash to decrease around the oval track, it would need to be on the other side of the track, anticipating the dog's return to start.

As the dog runs around the track clockwise from the bottom, the leash follows, creating a left half oval. As the dog returns to the bottom, the leash must anticipate the dog's path, tracing the right half oval as the leash length decreases as the dog returns to the start. But in order for that to occur, at the middle, the leash must switch from lying along the left half oval to lying along the right half oval. This will require the instantaneous travel of the leash from the left half to the right half, a physical impossibility.

<u>The math determines the situation</u>

In mathematical modeling, the situation is supposed to dictate the variables, the formulas, and the relationships between them. In Kim's approach to the box problem, It was the mathematics that dictated her image of the situation.

This is a related phenomenon to "problem before mental image" and was partially discussed there. Recall the following excerpt:

[176][331.435 Carlos] You know, in this diagram, the length isn't changing?
[177][334.149 Carlos] but in the problem you did in class, the length was changing

***** KIM *****
[178][336.791 Kim] Oh, cause we were told to change the length

In line 178, Kim's justification for "the length" changing is not based in the situation of the problem, but rather because she was told to change the length. Because she is not imagining the situation (she doesn't know how the length changed), she must be imaging that the value of the length changes. That is, that the length is a variable, and the number associated with that variable changes (because she was told to change the length), so the length changes. She does not imagine “the length” as a result of a measurement process occurring within the world of her mental image. ,

An even more explicit example of the mathematics determining the situation is found in an earlier excerpt:

[137][264.501 Kim] If there's a formula involving the length
[138][266.723 Kim] then it would vary the volume
[139][268.977 Kim]
[140][269.748 Kim] I don't know what the formula is though

Here, Kim explicitly states (lines 137-138) that the relationship between “the length” and “the volume” dependent on the existence of a mathematical formula. If there is a formula that relates “the volume” to “the length”, then the changing ”the length” changes “the volume”. Without that formula, either changing “the length” does not change “the volume”, or the relationship between “the length” and “the volume” is indeterminate (depending on Kim's meaning of if). In either case, the actual problem situation does not determine whether or not length changes volume, the formula does. One must imagine that Kim is willing to change her mental image of the situation based on whether or not she finds (remembers or is told) the formula for the volume of a box. Again, Kim is treating “the length” and “the volume” as variables representing values rather than as quantities resulting from measurement. The relationship between these variables is defined by a formula, not by a mental image.

A third example of this phenomenon comes from the actual construction of the formula for the volume as a function of the length of the side of the cutout:

[381][728.778 Kim] 11 minus your cutout
[382][730.864 Kim] oh, I remember this now, kind of
[383][733.311 Kim] and then I think you times that by
[384][735.853 Kim] 8.5
[385][738.857 Kim] I'm trying to remember
[386][740.275 Kim]
[387][741.813 Kim] I think and then your cutout
[388][743.599 Kim] like 8.5 minus your cutout
[389][746.146 Kim]
[390][748.425 Kim] and then
[391][750.135 Kim] I don't know what the height is

Here, Ali reconstructs the formula from memory (lines 382, 385), not from imaging the process of cutting out and folding the box. however, she does not remember the formula for the height.

[396][763.523 Kim] i don't know what the height would be
[397][764.499 Kim]

***** CARLOS *****
[398][765.373 Carlos] could you

***** KIM *****
[399][765.733 Kim] maybe the size of the cuout
[400][767.142 Kim] I think

***** CARLOS *****
[401][767.991 Carlos] ok
[402][768.985 Carlos] why do you think the size of the cutout?

[403][770.6 Carlos] you weren't sure before

***** KIM *****
[404][772.584 Kim] I don't know, just
[405][773.727 Kim] kind of guessing

***** CARLOS *****
[406][775.475 Carlos] ok, well
[407][776.775 Carlos]

***** KIM *****
[408][780.546 Kim] also -- well maybe because when you take out a corner
[409][783.909 Kim] when you end up folding it, that's kinda your height

In line [399], Kim remembered the formula for the height, and then on lines 408-409, Kim provided a justification from the situation that the height is the length of the side of the cutout. However, that justification is *a posteriori*. The mental image is used to justify a formula that she already knew, it is not guiding the creation of a new formula (line 404-405). I conjecture that had she remembered a different formula, she would have adjusted (within limits) her mental image to accommodate the formula, and not vice-versa.

Covariation

Mental Action 2

Mental Action 2 is the act of coordinating directions of change in two quantities. Because Kim rarely engaged in quantification as a process of measurement, it would be inaccurate to say that she engaged in thinking about two quantities varying simultaneously. Rather than discuss her covariation of quantities, we can discuss her covariation of values – that is, imagining the values of two variables changing simultaneously, while those variables represent abstract numbers rather than results of measurements. Kim would often talk about values “increasing” or “decreasing,” but with the exception of the fox and greyhound problem (discussed in a later section), never talked about amounts of change. That is, she never spoke in a way that indicated that she was making use of mental action 3:

[489][954.703 Kim] it's 1.8 and then 1.9's right here
[490][957.46 Kim] so as the cutout varies
[491][959.89 Kim] from 1.8 inches to 1.9 inches it
[492][962.117 Kim] the length
[493][963.625 Kim]
[494][965.09 Kim] the volume decreases
[495][966.769 Kim]

In this excerpt, Kim was working from a graph on gridded paper. She certainly had the materials to estimate the amount that “the volume” has changed. In class, the phrasing for referring to amounts of change has been emphasized in class and on exams: “As the length of the side of the cutout increases from 1.8 inches to 1.9 inches, the volume decreases from 65.27 inches cubed to 64.30 inches cubed.” But Kim did not see a need to talk about amounts. She considered “the volume is decreasing” to be a sufficient answer, indicating that she thought about this problem in terms of directions, but not in terms of amounts of change.

This excerpt provides further evidence that Kim was not thinking about quantities, but rather as values. 12.8 seconds later, Kim began to rework her solution to task one based an a confusion between “the length” of the side of the cutout and “the length” of the side of the box:

***** KIM *****
[502][977.846 Kim] so that's [task 1] wrong.
[503][980.262 Kim] right?

***** -CARLOS-KIM *****
[504][981.127 1] Why is that wrong
[---][981.127 2] as the length

***** KIM *****
[505][982.411 Kim] increases, the volume
[506][984.496 Kim]
[507][985.648 Kim] cause the length is increasing here
[508][987.971 Kim] so then the volume decreases
[509][990.193 Kim] so this would be the opposite

Single Variable Varying

Although Kim did occasionally engage in Mental Action 2 on values, as in the above example, this mental action was rare for her. One way in which Kim did not engage in thinking about two variables changing simultaneously is that Kim often described the change in a value as occurring without reference to a second value changing, I include an example from the train problem below:

[1138][2309.924 Kim] so as
[1139][2310.797 Kim] as he moves the distance will
[1140][2312.751 Kim] change
[1141][2313.676 Kim] cause he's moving along the train which is a mile and a half.

In lines 1193-1140, Kim described the distance changing, but she never describes the distance changing as a second variable (time) changes. This fact that the second variable here is time, and Kim very rarely used time explicitly.

Local Means Global

In the following excerpt, Kim rethought her solution to Task 1. Prior to this excerpt, she concluded that as “the length” decreases, “the volume” would always decrease.

***** -CARLOS-KIM *****
[504][981.127 1] Why is that wrong
[---][981.127 2] as the length

***** KIM *****
[505][982.411 Kim] increases, the volume
[506][984.496 Kim]
[507][985.648 Kim] cause the length is increasing here
[508][987.971 Kim] so then the volume decreases
[509][990.193 Kim] so this would be the opposite

***** CARLOS *****
[510][991.539 Carlos] so as the length increases, the volume decreases?

***** KIM *****
[511][994.751 Kim] mhm

***** CARLOS *****
[512][995.162 Carlos] ok

***** -KIM-CARLOS *****
[513][995.881 1] cause the covary
[---][995.881 2] will that al-

***** CARLOS *****
[514][996.929 Carlos] will that always be true?
[515][998.184 Carlos]

***** KIM *****
[516][998.955 Kim] oh, yeah

***** CARLOS *****
[517][999.965 Carlos] well
[518][1001.159 Carlos] you-you-you thought this was true to begin with

***** KIM *****
[519][1005.476 Kim] uhhuh

***** CARLOS *****

[520][1006.209 Carlos] and now you think that the opposite is true?

***** KIM *****
[521][1009.066 Kim] well, yeah, well looking at the graph

In lines 507-507, Kim reached a new conclusion based on the graph: that as “the length” increases, “the volume” decreases; which is the opposite conclusion to what she reached earlier (line 508). She agreed that this new conclusion (over the entire domain) on lines 511, 516, and 521, and justified her new conclusion on the graph (line 521). However, the graph that was provided to her was a graph over the entire domain of the problem. The graph showed both the volume of the box increasing as the length of the side of the cutout decreased, and as the length of the side of the cutout increased further, the volume began to decrease again. Despite the fact that the graph was ‘hump shaped,” Kim based her conclusion entirely on the small portion that she had focused on for task 3, and used the solution from task 3 to “disprove” to her self her original conclusion over the entire domain of “the length.”

This indicates that here, although Kim was using language such as “increasing” and “decreasing,” Kim was not imagining how the directions of the values changed; nor did she imagine how they changed in tandem. In short, Kim was not covarying two values.

Mental Action 3

The only problem in which Kim engaged in covarying amounts of change was the Fox and Greyhound problem:

[1920][3966.486 Kim] yeah
[1921][3968.749 Kim] that's why I'm being confused, because like fraction
[1922][3972.091 Kim] but
[1923][3973.684 Kim] so if the
[1924][3976.1 Kim] greyhound's going seven
[1925][3977.623 Kim] leaps
[1926][3979.679 Kim]
[1927][3981.427 Kim] it equals
[1928][3982.762 Kim] 10 plus five thirds
[1929][3984.612 Kim] fox leaps
[1930][3985.692 Kim]
[1931][3991.808 Kim]

Here, Kim implicitly quantified time and distance for the greyhound by converting the distance the ground traveled to units of fox leaps (line 1928-1929). She then used increments of seven greyhound leaps as units of time, and covaried the amount of distance that the greyhound traveled (measured in fox leaps) with the amount of time that the chase went on (measured in units of 7 greyhound leaps leaps):

[1958][4047.063 Kim] if it took another seven
[1959][4048.605 Kim] leaps it's be at
[1960][4050.713 Kim]
[1961][4055.031 Kim] ten thirds?

***** CARLOS *****
[1962][4057.498 Carlos] mhm

***** KIM *****
[1963][4058.629 Kim] plus 20

In line 1958-1989, Kim demonstrated her use of seven greyhound leaps as a unit of time, and on lines 1959-1961 and 1963, Kim demonstrated her use of fox leaps as a unit of distance. Kim uses these units as a guide for covarying greyhound distance and time. In lines 1924-1929 above, she stated that after 7 greyhound leaps of time, the greyhound traveled 10 + 5/3 fox leaps of distance. And in lines 1958-1963 Kim performs an additional calculation: that after another 7 greyhound leaps of time, the greyhound will have traveled another 10 + 5/3 fox leaps of distance, for a total of 20 + 10/3 fox leaps of distance traveled. Although she does not articulate this so explicitly, she continues her pattern of adding seven greyhound leaps and adding 10 + 5/3 fox leaps through the end of the tape:

[2104][4384.738 Kim] so thirteen
[2105][4387.976 Kim] would be 140
[2106][4389.569 Kim] + 5/3

On line 2104, Kim has now added together 13 units of 7 greyhound leaps, one at a time, and found a total distance for the greyhound of 140 + 5/3 fox leaps at 13 seven greyhound leaps of time. She achieved this not by multiplication, but by adding each unit of time and corresponding amount of change of distance individually.

This is not to say that Kim was incapable of a multiplicative solution. She covaried amounts of change of time measured in greyhound leaps and amounts of change of fox leaps multiplicatively:

[2015][4185.589 Kim] I think that was six, so seven
[2016][4188.415 Kim]
[2017][4190.933 Kim] leaps he'd be at
[2018][4192.578 Kim] 80 plus
[2019][4195.148 Kim] five thirds
[2020][4198.901 Kim] and then
[2021][4201.778 Kim]
[2022][4203.783 Kim] the fox would be at
[2023][4205.673 Kim] seven times 9

In line 2015, Kim indicates that we are in the seventh group of seven greyhound leaps. On line 2017, she calls this group seven leaps, but she corrects herself later in line 2068 below. She determines the distance the fox has traveled multiplicatively on line 2023.

So although Kim is capable of reasoning about this problem multiplicatively, she chose to calculate greyhound distance additively. I attribute this phenomena to a general discomfort with fractions. Kim can calculate the distance the fox has traveled using units of seven greyhound leaps without multiplying fractions, but the same is not true of the greyhound's distance. Kim expresses this discomfort on line 1921.

[1921][3968.749 Kim] that's why I'm being confused, because like fraction

In closing this section, it is important to note that when Kim called seven groups of seven leaps "seven leaps" on lines 2015-2017 above, that this was an error of speaking, not an error in the way that she was thinking about the problem. On lines 2068-2072 below, she corrects herself without prompting:

[2068][4313.516 Kim] 10 leaps
[2069][4314.237 Kim] err ten
[2070][4315.265 Kim]
[2071][4317.66 Kim] not ten leaps but
[2072][4319.254 Kim] ten like seven leaps

On line 2072, Kim uses the phrase "seven leaps" as a unit, indicating that she is indeed thinking about quantifying time in units of seven greyhound leaps, rather than units of greyhound leaps.

Discussion - Conclusions

Quantification

Kim difficulties in solving these problems were rooted in her difficulties with quantification, imagined measurement, and developing a mental image of the situation, indicating that these are necessary components of orienting to these problems mathematically. While I do not yet propose a model of an orienting phase of problem solving, it is clear that forming a mental image of the problem situation and quantifying within that mental image must be a part of the orienting process.

Time appeared to be particularly difficult for Kim to quantify. I attribute this to the abstract nature of time. There is no visible object to imagine measuring: at best one can visualize a measuring device such as a stopwatch, or a visual spatial metaphor such as distance traveled along a path or the length between point events.

In summary, Kim's difficulties demonstrated three necessary components to the orienting phase of problem solving: forming a mental image of the situation, selecting quantities, and imagining those quantities as acts of measurement occurring in the mental image. Future studies might focus on the development of a model of orientation, refining the models of quantification and measurement described here, or further exploring the quantification of time.

Mental Action 3

In the first three problems given, Kim relied exclusively on mental action 2 -- coordinating the direction of quantities or values. However, in the fox and the greyhound problem, Kim made use of amounts of change, indicating both the ability and the need to use covariation mental action 3.

I attribute the ability of Kim to use mental action 3 to her quantification of time in the fox and greyhound problem. With time as a quantified variable, Kim had access to two changing quantities, with salient numerical values, which she could compare. Compare this to the Train problem, the other problem in which Kim discussed time. However, In the train problem, Kim did not quantify time as a changing quantity, but only as a 15 minute interval. Without two changing quantities, Kim was forced to rely on the "single variable varying" descriptions described above.

In the box problem, Kim was given two quantities: length of the side of the cutout, and volume of the box. During task 3, both of these quantities had numerical values provided in the form of the graph. However, although Kim had the tools necessary to engage in mental action 3, she was satisfied with mental action 2. She did not have an *intellectual need* to engage in mental action 3 (Harel, in press).

Harel use the idea of an intellectual need as a prerequisite for learning, but I apply it here as an explanation of Kim's behaviour. Harel describes an *intellectual need* as "a behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate"

In the Fox and Greyhound problem, Kim encountered the problem of faction multiplication, a problem she was not comfortable with. However, she was willing to work with fractions additively. This additive approach generated an *intellectual need* to keep track of how many times she had added, which lead to the quantification of time, and the coordination of amounts of changes in time and changes in distance. Amounts of changes in time in units of seven greyhound leaps was a ay of keep track of how many times she had added, and the distance traveled by the greyhound measured in fox leaps was a way of keeping track of how much she had added. The amounts of change of distance in greyhound leaps coordinated the changes in time and the total distance traveled.

So Kim used mental action 3 in the fox and greyhound problem both because she was capable of coordinating two quantities (distance and time, measured in units of leaps), and because she had a need to keep track of amounts of change, so that she could approach the problem additively.

Future work here might involve the study of impact of quantification of two variables as a necessary component to mental action 3, the role of time in covariation of two quantities, or a focus on fostering intellectual need for students to learn and use amounts of change when covarying quantities.

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